BB Radiation- Tutorial 1 Solutions

14 December 2021 00:05

(1)

1. * According to Planck, the spectral energy density $u(\lambda)$ of a blackbody maintained at temperature T is given by

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}$$

where λ denotes the wavelength of radiation emitted by the blackbody.

(a) Find an expression for $\lambda_{\max}$ at which $u(\lambda, T)$ attains its maximum value (at a fixed temperature T). $\lambda_{\max}$ should be in terms of T and fundamental constants h, c and k_B.

(b) Expressing $\lambda_{\max}$ as $\frac{\alpha}{T}$, obtain an expression for $u_{\max}(T)$ in terms of α, T and the fundamental constants.

(a) $U(\lambda, t)$ attains maximum when $\left.\frac{dU(\lambda, T)}{d\lambda}\right|_{\lambda = \lambda_{max}} = 0$

$$8\pi hc = A \qquad \frac{hc}{k_B T} = B$$

$$\frac{d}{d\lambda}\left(\frac{A}{\lambda^5} \cdot \frac{1}{\exp\left(\frac{B}{\lambda} - 1\right)}\right) = 0$$

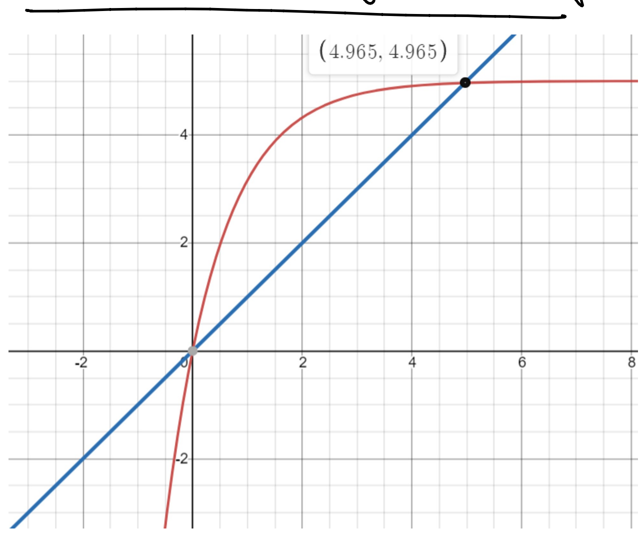
$$-\frac{5A}{\lambda^6}\frac{1}{e^{B/\lambda} - 1} + \frac{A}{\lambda^5} \cdot \frac{e^{B/\lambda}}{(e^{B/\lambda} - 1)^2}\left(\frac{B}{\lambda^2}\right) = 0$$

$$\Rightarrow \quad 5\lambda = \frac{e^{B/\lambda} \cdot B}{(e^{B/\lambda} - 1)} \qquad \left(\because \begin{array}{l} A, \lambda \neq 0 \\ \lambda \neq -\ln B \end{array}\right)$$

$$\Rightarrow \quad 5\left(1 - e^{-B/\lambda}\right) = \frac{B}{\lambda}$$

$\downarrow$

① Can be solved graphically



$$\frac{B}{\lambda} = 4.965$$

$$\lambda = \left(\frac{hc}{k_B T}\right)\left(\frac{1}{4.965}\right)$$

$$= \frac{2898.9 \times 10^{-6}}{T}$$

$$\left\{k_B = 1.38 \times 10^{-23}\right\}$$

② Can also be solved numerically

Take $\frac{B}{\lambda} = 5 - t$

$$\Rightarrow \quad 5e^{-(5-t)} = t$$

that $t \simeq 0$)

$$\Rightarrow \quad 5e^{-(5-t)} = t$$

(By drawing rough graphs we can have a rough idea that $t \simeq 0$)

So for now we can approximate

$\underline{t \simeq 5e^{-5}} = \underline{0.033}$ (also justifies our assumption)

$\frac{B}{h} = 5 - 0.033 = 4.967$ (Quit close to the actual value)

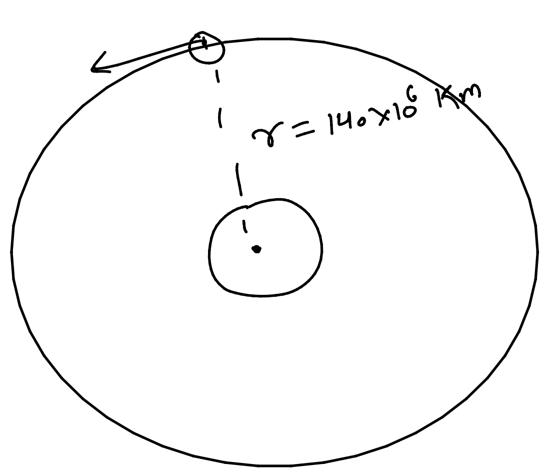
(b) $\lambda_{max} = \alpha / T$

$$U_{max}(T) = \frac{8\pi h c T^5}{\alpha^5} \frac{1}{e^{hc/\alpha K_b} - 1}$$

(2)

2. The earth rotates in a circular orbit about the sun. The radius of the orbit is 140×10^6 km. The radius of the earth is 6000 km and the radius of the sun is 700,000 km. The surface temperature of the sun is 6000 K. Assuming that the sun and the earth are perfect black bodies, calculate the equilibrium temperature of the earth.

Sol^n



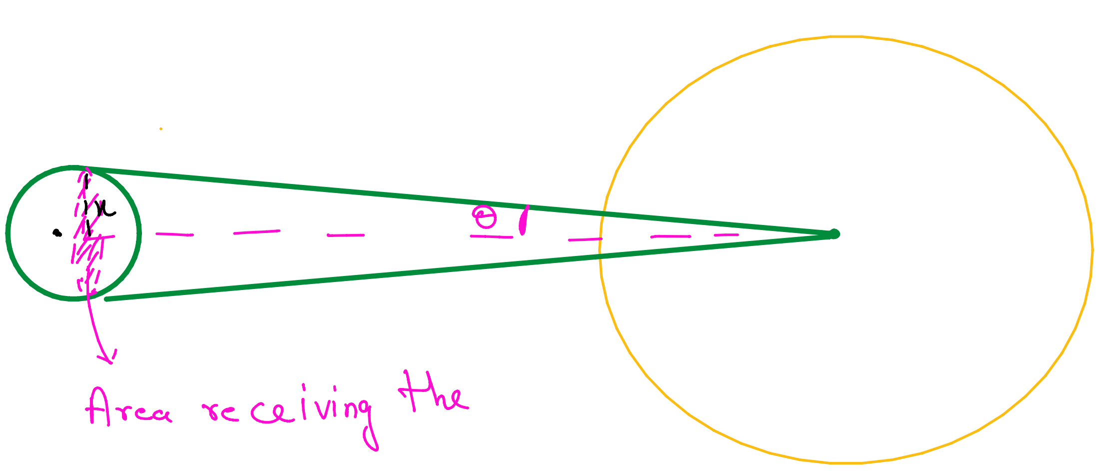
Total Power radiated by sun

$$P_o = \sigma A_s T_s^4$$

Intensity at earth's location

$$= \frac{P_o}{4\pi r^2}$$

$\left\{ \because \text{Since Power is radiated uniformly in a sphere of radius } r \right\}$

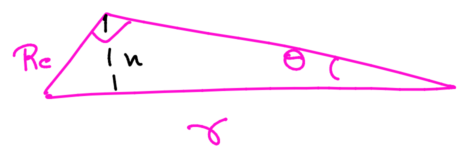


Power:

$$= \pi h^2$$

$$= \pi \left(R_e \cos\theta\right)^2$$

$$\simeq \pi R_e^2$$



$$\sin\theta = \frac{R_e}{R} \simeq 0$$

Power received by Earth

$$= (\text{Intensity}) \times (\text{Cross section area})$$

$$= \frac{P_0}{4\pi r^2} \times \left(\pi R_e^2\right) = \frac{P_0 R_e^2}{4 r^2}$$

$P_e =$ Power radiated by Earth $= \sigma A_e T_e^4 = \sigma\left(4\pi R_e^2\right) T_e^4$

In equilibrium

Power received by earth = Power radiated by Earth

$$\frac{P_0 R_e^2}{4 r^2} = \sigma\left(4\pi R_e^2\right) T_e^4$$

$$\sigma \cancel{\left(4\right.}\pi R_s^2\left.\right) T_s^4 \left(\frac{\cancel{R_e^2}}{4 r^2}\right) = \sigma \cancel{\left(4\pi R_e^2\right)} T_e^4$$

$$T_e = T_s \sqrt{\frac{R_s}{2r}} = 300\,K$$

(3)

3. (a) Given Planck's formula for the energy density, obtain an expression for the Rayleigh Jeans formula for $U(\nu, T)$.

(b) For a black body at temperature T, $U(\nu, T)$ was measured at $\nu = \nu_0$. This value is found to be one tenth of the value estimated using Rayleigh Jeans formula. Obtain an implicit equation in terms of $h\nu/k_B T$

(c) Solve the above equation to obtain the value of $h\nu/k_B T$, up to the first decimal place.

Sol^n

(a) Rayleigh- Jeans Law $\quad U_J(\nu, T) = \dfrac{8\pi\nu^2}{c^3} k_B T$

Planck's Law $\quad U_P(\nu, T) = \dfrac{8\pi\nu^2}{c^3} \dfrac{h\nu}{e^{h\nu/k_B T} - 1}$

Rayleigh- Jeans law at lower frequency

Planck's law takes form of Rayleigh-Jeans law at lower frequency value.

$$U_P(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_BT} - 1} \simeq \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\left(1 + \frac{h\nu}{k_BT}\right) - 1} \simeq \frac{8\pi\nu^2}{c^3} k_BT = U_J(\nu, T)$$

(b) $U_P(\nu_0, T) = \frac{1}{10} U_J(\nu_0, T)$

$$\frac{\cancel{8\pi\nu_0^2}}{\cancel{c^3}} \frac{h\nu_0}{e^{h\nu_0/k_BT} - 1} = \frac{\cancel{8\pi\nu_0^2}}{\cancel{c^3}} k_BT \left(\frac{1}{10}\right)$$

$$10\left(\frac{h\nu_0}{k_BT}\right) = e^{h\nu_0/k_BT} - 1$$

(c) $\frac{h\nu_0}{k_BT} = \alpha$ $\qquad 10\alpha = e^{\alpha} - 1$

① Either solve using desmos

② Using Calculator

$$10\alpha + 1 = e^{\alpha}$$

α	$10\alpha + 1$	e^{α}
1	11	2.73
3	31	20.85
4	41	54.59
3.5	36	33.11
3.7	38	40.44
3.6	37	36.59

$\frac{h\nu}{k_BT} = 3.6$ (upto first decimal place)

④ 4. Using appropriate approximations, derive Weins' displacement law from Planck's formula for energy density of black body radiation.

... obtained the wavelength λ_{max} corresponding

Sol^n In solution of question 1, we obtained the wavelength λ_{max} corresponding to maximum $u(\lambda, T)$.

Upon solving we get $\lambda_{max} = \frac{\alpha}{T}$ (α = const.)

$$\boxed{\lambda_{max} \cdot T = \alpha = \text{Constant}}$$