Tutorial 4 Solutions Fourier Transform

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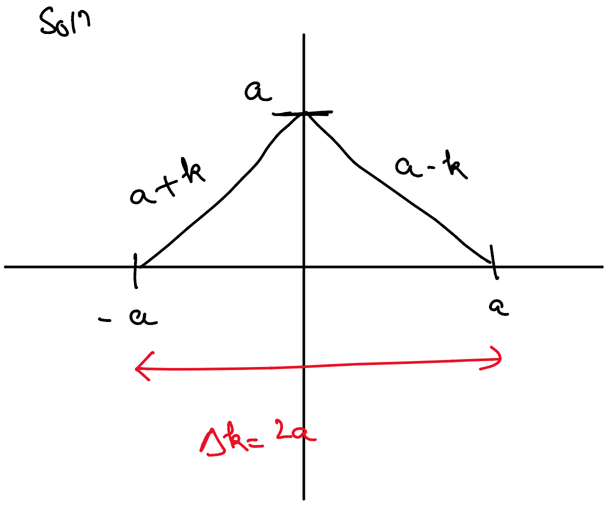
1. *If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
(a) Find the Fourier transform for $\phi(k)$
(b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

Soln



$$\int_{-\infty}^{\infty} \phi(k) \cdot \phi^*(k) \cdot dk = 1 \qquad \{ \because \phi(k) \text{ is normalized} \}$$

$$\Rightarrow \int_{-a}^{0} A^2 (a+k)^2 \cdot dk + \int_{0}^{a} A^2 (a-k)^2 \cdot dk = 1$$

$$\Rightarrow \frac{A^2}{3}(a^3) + \frac{A^2}{3}(a^3) = 1$$

$$\Rightarrow \frac{2A^2}{3} a^3 = 1$$

$$\boxed{A = \pm \sqrt{\frac{3}{2} a^3}}$$ (Usually taken as +ve one)

(a) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \phi(k) e^{ikx} \cdot dk = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} (k+a) e^{ikx} \cdot dk + \int_{0}^{a} (a-k) e^{ikx} dk \right]$$

Subs. $k = -k$

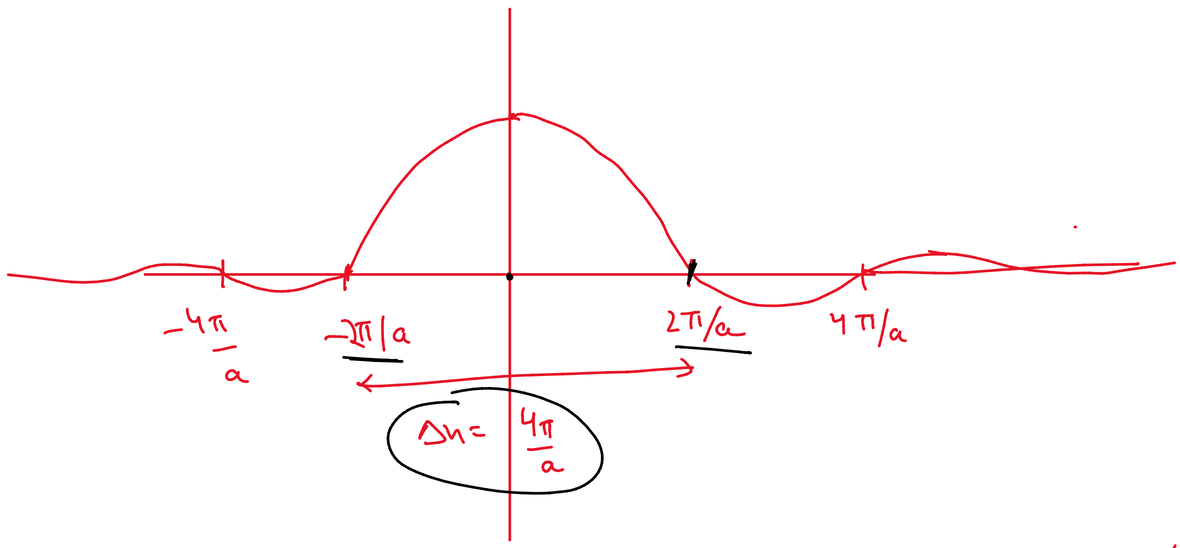
$$\hookrightarrow \int_{a}^{0} (-k+a) e^{-ikx} (-dk) = \int_{0}^{a} (a-k) e^{-ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{a} (a-k) \left(e^{ikx} + e^{-ikx}\right) \right] \cdot dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} 2(a-k) \cos kx \cdot dk$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(a-k)}{x} \sin kx \Big|_0^a + \int_0^a \frac{\sin kx}{x} \cdot dk \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(1 - \cos ax)}{x^2} \right]$$



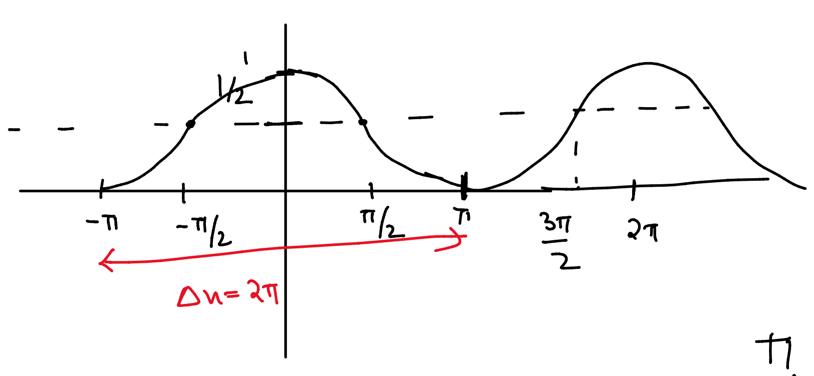
$P = \hbar k \Rightarrow \Delta p = \hbar \Delta k$

$\Delta x \cdot \Delta k = \frac{4\pi}{a} \cdot 2a = 8\pi = \Delta x \cdot \frac{\Delta p}{\hbar}$

$\boxed{\Delta x \cdot \Delta p = 8\pi\hbar}$

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)$ $(for -\pi \le x \le \pi)$ and $f(x) = 0$ elsewhere
   (a) Plot $f(x)$ versus x.
   (b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
   (c) At what value of k, $|g(k)|$ attains its maximum value?
   (d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
   (e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

(a)



$\cos^2(x/2) = \frac{1}{2}(1+\cos x)$

Sol^n (b) $g(k) = \int_{-\pi}^{\pi} \cos^2\left(\frac{x}{2}\right) e^{-ikx}\,dx = \int_0^{\pi} \cos^2\frac{x}{2}\left(e^{-ikx} + e^{ikx}\right) dx$

$= \int_0^{\pi} \frac{(1+\cos x)}{\cancel{2}} (\cancel{2}\cos kx)\,dx$

$= \int_0^{\pi} \cos kx + \frac{1}{2}(\cos(k+1)x + \cos(k-1)x)\,dx$

$= \frac{\sin k\pi}{k} + \frac{\sin(k+1)\pi}{2(k+1)} + \frac{\sin(k-1)\pi}{2(k-1)}$

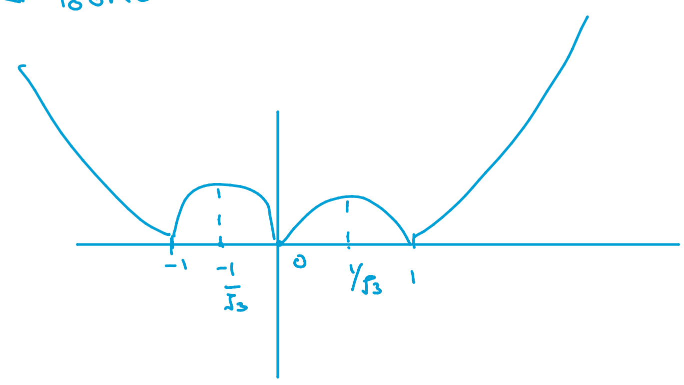
$= \frac{\sin k\pi}{k} - \frac{1}{2}\left[\frac{\sin k\pi}{(k+1)} + \frac{\sin k\pi}{(k-1)}\right]$

$= \sin k\pi \left[\frac{(k^2-1) - (k^2)}{k(k^2-1)}\right]$

$= \frac{-\sin k\pi}{k(k^2-1)} = \frac{\sin(k\pi)}{k - k^3} = g(k)$

(c) $|g(k)| = \left|\frac{\sin(k\pi)}{k - k^3}\right| = \frac{|\sin k\pi|}{|k - k^3|}$

(c) $|g(k)| = \left| \dfrac{\sin(k\pi)}{k - k^3} \right| = \dfrac{|\sin \ldots|}{(|k - k^3|)}$ looks like this

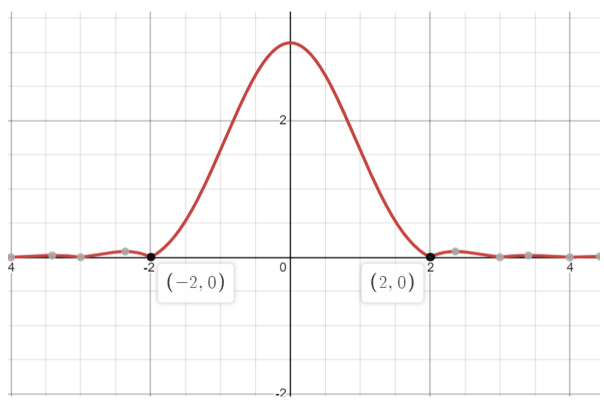


since $|k - k^3|$ i

$$\lim_{k \to 0} \frac{-\sin k\pi}{k(k-1)(k+1)} = +\pi$$

$$\lim_{k \to 1} - \frac{\sin k\pi}{k(k-1)(k+1)} = \frac{\sin(k-1)\pi}{k(k-1)(k+1)} = +\frac{\pi}{2}$$

$$\lim_{k \to -1} - \frac{\sin k\pi}{k(k-1)(k+1)} = \frac{+\sin(k+1)\pi}{k(k-1)(k+1)} = +\frac{\pi}{2}$$



(d) Zeroes when $\underline{\sin k\pi = 0}$ $(k \neq 0, 1, -1)$

$\Rightarrow k = \pm 2, \pm 3 \ldots$

$\Delta k = 4$

$\Delta x \cdot \Delta k = 8\pi$

Particle in a box

3. *A wave function $\psi(x)$ is defined such that $\psi(x) = \sqrt{2/L}\sin(\pi x/L)$ for $0 \le x \le L$ and $\psi(x) = 0$ otherwise.
(a) Writing $\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$, find $a(k)$.
(b) What is the amplitude of the plane wave of wavelength L constituting $\psi(x)$?

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) e^{ikx} \, dk$$

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} \, dx$$

Sol^n $a(k) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \psi(x) e^{-ikx} \, dx$

$$= \frac{1}{2\pi} \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-ikx} \, dx = \frac{1}{2\pi} \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \left(\cos kx - i \sin(kx)\right) dx$$

$$= \frac{1}{2\pi} \sqrt{\frac{2}{L}} \int_0^L \frac{1}{2}\left(\sin\left(\frac{\pi}{L}+k\right)x\right) + \frac{1}{2}\left(\sin\left(\frac{\pi}{L}-k\right)x\right) \cdot dx$$

$$+ \frac{1}{2\pi} \sqrt{\frac{2}{L}} (-i) \int_0^L \frac{1}{2}\cos\left(\frac{\pi}{L}-k\right)x - \frac{1}{2}\cos\left(\frac{\pi}{L}+k\right)x \, . dx$$

$$= \frac{1}{2\pi} \sqrt{\frac{2}{L}} \, \frac{1}{2} \left[\frac{1 + \cos kL}{\frac{\pi}{L} + k} + \frac{1 + \cos kL}{\frac{\pi}{L} - k} \right] - \frac{i}{2\pi} \sqrt{\frac{2}{L}} \cdot \frac{1}{2} \left[\frac{\sin kL}{\frac{\pi}{L} - k} + \frac{\sin kL}{\frac{\pi}{L} + k} \right]$$

$$\frac{1}{\ } \left[\left[\frac{(1 + \cos kL)(2\pi)}{\ldots} \right] - i \left[\frac{(\sin kL)(2\pi)}{\ldots} \right] \right.$$

$$= \frac{1}{2\pi}\sqrt{\frac{L}{2}}\left[\frac{(1+\cos kL)(2\pi)}{(\pi^2-k^2L^2)}\right] - i\left[\frac{(\sin kL)(2\pi)}{(\pi^2-k^2L^2)}\right]$$

$$= \sqrt{\frac{L}{2}}\,\frac{1}{(\pi^2-k^2L^2)}\left[(1+\cos kL) - i\,(\sin kL)\right]$$

corresponding amplitude

(b) $f(x) \longleftrightarrow \sum_k a(k)\, e^{ikx}$

Plane wave with wave number k

$$a\left(k=\frac{2\pi}{L}\right) = \sqrt{\frac{L}{2}}\left(\frac{1}{-3\pi^2}\right)(2)$$