Tutorial 5 Solutions

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↳ Trailer of MA106

Inner product → $\langle \varphi_1, \varphi_2 \rangle = \int_{-\infty}^{\infty} \varphi_1^* \varphi_2 \, dx$ ($\varphi_1^* \to \varphi_1^\dagger$)

$\varphi_1 = \varphi_2$ $\quad \langle \varphi_1, \varphi_1 \rangle = |\varphi_1|^2$

$\left(\frac{\partial}{\partial x}\right)\varphi = \frac{\partial \varphi}{\partial x}$

Adjoint of an operator → $\hat{O} f = g$

$\langle \varphi_1, \hat{A}\varphi_2 \rangle$

$\langle \hat{A}\varphi_1, \varphi_2 \rangle = \int (\hat{A}\varphi_1)^* \varphi_2$ $\qquad \int \varphi_1^* (\hat{B}\varphi_2) = \langle \varphi_1, \hat{B}\varphi_2 \rangle$ $\qquad \hat{B} = (\hat{A})^\dagger$

It can be proved that this inner product can be expressed in this form for an operator $\hat{B}$

then $\hat{B}$ is defined as $\hat{A}^*$

For a hermitian operator $\hat{B} = \hat{A}$ or $(\hat{A}^* = \hat{A})$

1. Which of the operators A_i defined in the following are linear operators? Which of these are hermitian? All the functions $\psi(x)$ are well behaved functions vanishing at $\pm\infty$.
   (a) $\hat{A}_1\psi(x) = \psi(x)^2$
   (b) $\hat{A}_2\psi(x) = \frac{\partial \psi(x)}{\partial x}$
   (c) $\hat{A}_3\psi(x) = \int_a^x \psi(x')\,dx'$
   (d) $\hat{A}_4\psi(x) = 1/\psi(x)$
   (e) $\hat{A}_5\psi(x) = -\psi(x+a)$
   (f) $\hat{A}_6\psi(x) = \sin(\psi(x))$
   (g) $\hat{A}_7\psi(x) = \frac{\partial^2 \psi(x)}{\partial x^2}$

$\hat{A}(a\varphi_1 + b\varphi_2) = a\hat{A}\varphi_1 + b\hat{A}\varphi_2$ $\qquad \forall\, a, b \in \mathbb{C}$

Linear Operator $\qquad \hat{A}(\varphi_1 + \varphi_2) = \hat{A}\varphi_1 + \hat{A}\varphi_2$

(a) $\hat{A}_1(\varphi_1 + \varphi_2) = (\varphi_1 + \varphi_2)^2 \neq \varphi_1^2 + \varphi_2^2 = \hat{A}\varphi_1 + \hat{A}\varphi_2$ → Non linear

$\int (\hat{A}_1\varphi_1)^* \varphi_2 = \int (\varphi_1^2)^* \varphi_2$

$\int \varphi_1^* \hat{A}\varphi_2 = \int \varphi_1^* \varphi_2^2$

Non hermitian

(b) **Linear** $\qquad \frac{\partial}{\partial x}(\varphi_1 + \varphi_2) = \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial x} = \frac{\partial}{\partial x}(\varphi_1) + \frac{\partial}{\partial x}(\varphi_2)$

$\langle \hat{A}_2\varphi_1, \varphi_2 \rangle$ $\qquad \int (\hat{A}_2\varphi_1)^* \varphi_2 = \int \left(\frac{\partial \varphi_1}{\partial x}\right)^* \varphi_2$

(why?)

$\langle \varphi_1, \hat{A}_2\varphi_2 \rangle$ $\qquad \int \varphi_1^* \hat{A}_2\varphi_2 = \int_{-\infty}^{\infty} \varphi_1^* \frac{\partial \varphi_2}{\partial x} = \cancel{\varphi_1^* \varphi_2 \Big|_{-\infty}^{\infty}}^{\,0} - \int \varphi_2 \frac{\partial \varphi_1^*}{\partial x}$

Anti-hermitian

(c) $\hat{A}_3\psi(x) = \int_a^x \varphi(x')\,dx$ → linear

similar method as above

(c) $\hat{A}_3 \psi(x) = \int_a^x \varphi(x')\,dx' \longrightarrow$ linear

Check if hermitian using similar method as above

(d) $\rightarrow$ Non linear

$$\rightarrow \int (\hat{A}_4 \varphi(x))^* \varphi_2 = \int \frac{\varphi_2}{\varphi_1^*}$$

$$\int \varphi_1^* \hat{A} \varphi_2 = \int \frac{\varphi_1^*}{\varphi_2}$$

Non hermitian

(f) $\hat{A}_6 \psi(x) = \sin \varphi(x) \longrightarrow$ Non linear

$$\int (\hat{A}_6 \varphi_1)^* \varphi_2 = \int \sin^*(\varphi_1) \cdot \varphi_2$$

$$\int \varphi_1^* (A_6 \varphi_2) = \int \varphi_1^* \sin \varphi_2$$

Non hermitian

(g) $\hat{A}_7 \varphi(x) = \dfrac{\partial^2 \varphi(x)}{\partial x^2} \longrightarrow$ Linear

$$\langle \hat{A}_7 \varphi_1, \varphi_2 \rangle = \int (\hat{A}_7 \varphi_1)^* \varphi_2 = \int \frac{\partial^2 \varphi_1^*}{\partial x^2} \cdot \varphi_2 = \cancel{\frac{\partial \varphi_1^*}{\partial x} \cdot \varphi_2 \Big|_{-\infty}^{\infty}}^{\,0} - \int \frac{\partial \varphi_1^*}{\partial x} \cdot \frac{\partial \varphi_2}{\partial x}$$

$$\langle \varphi_1, \hat{A}_7 \varphi_2 \rangle = \int \varphi_1^* (\hat{A}_7 \varphi_2) = \int \varphi^* \frac{\partial^2 \varphi_2}{\partial x^2} = \cancel{\varphi_1^* \frac{\partial \varphi_2}{\partial x} \Big|_{-\infty}^{\infty}}^{\,0} - \int \frac{\partial \varphi_1^*}{\partial x} \cdot \frac{\partial \varphi_2}{\partial x}$$

hermitian

Can also be seen by observing that it is hamiltonian (upto a constant)

2. (a) If $\hat{A}$ and $\hat{B}$ are Hermitian and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$, prove that $\hat{C}$ is Hermitian

   (b) An operator is said to be anti-Hermitian if $\hat{O}^\dagger = -\hat{O}$. Prove that $[\hat{A}, \hat{B}]$ is anti-Hermitian.

Sol^n (a) Given

$$\int (\hat{A}\varphi_1)^* \varphi_2 = \int \varphi_1^* (\hat{A}\varphi_2) \quad - (1)$$

$$\int (\hat{B}\varphi_1)^* \varphi_2 = \int \varphi_1^* (\hat{B}\varphi_2) \quad - (2)$$

Consider an operator $\hat{C} = -i[\hat{A}, \hat{B}] = -i(\hat{A}\hat{B} - \hat{B}\hat{A})$

$$\int (\hat{C}\varphi_1)^* \varphi_2 = \int [-i(\hat{A}\hat{B} - \hat{B}\hat{A})\varphi_1]^* \varphi_2 = i \int \Big([\hat{A}\underbrace{\hat{B}\varphi_1}_{\varphi_1'}]^* - [\hat{B}\underbrace{\hat{A}\varphi_1}_{\varphi_1''}]^* \Big) \varphi_2$$

$$= i \int (\hat{B}\varphi_1)^* \hat{A}\varphi_2 - (\hat{A}\varphi_1)^* \hat{B}\varphi_2$$

$$= i \int \varphi_1^* \hat{B}\hat{A}\varphi_2 - \varphi_1^* \hat{A}\hat{B}\varphi_2$$

$$= -i \int \varphi_1^* (\hat{A}\hat{B} - \hat{B}\hat{A}) \varphi_2$$

$$= \int \varphi_1^* (\hat{C}\varphi_2)$$

Hence $\hat{C}$ is a hermitian operator

(b) DIY

(Extra − sign which appeared due to $i^* = -i$ won't appear)

3. * Prove that if $\hat{K}$ is a Hermitian operator, $\exp\left(i\hat{K}\right)$ is an unitary operator, and if $\hat{U}$ is an Unitary operator, then there is an operator K such that $\hat{U} = \exp\left(i\,\hat{K}\right)$, and this $\hat{K}$ is Hermitian.

Sol^n Unitary Operator → preserves the inner product for a unitary operator

i.e. $\langle \varphi_1, \varphi_2 \rangle = a \implies \langle \hat{U}\varphi_1, \hat{U}\varphi_2 \rangle = a$

$$\int (\hat{U}\varphi_1)^* (\hat{U}\varphi_2) = a$$

$$\int \varphi_1^* \hat{U}^* (\hat{U}\varphi_2) = \int \varphi_1^* (\hat{U}^* \hat{U})\, \varphi_2 = \int \varphi_1^* \varphi_2$$

$$\implies \boxed{\hat{U}^* \hat{U} = \hat{I}}$$

↳ identity operator

given $\hat{K} = \hat{K}^*$

Consider $\hat{U}^* \cdot \hat{U} = \left(e^{i\hat{K}}\right)^* \left(e^{i\hat{k}}\right) = e^{-i\hat{k}^*} \cdot e^{i\hat{k}} = e^{-i\hat{k}} \cdot e^{i\hat{k}} = e^{i(\hat{k}-\hat{k})}$

$= \hat{I}$

Hence $\hat{U}$ is unitary

(b) Start with an operator $\hat{p}$ & use $e^{i\hat{p}} = \hat{I} + i\hat{p} + \frac{(i\hat{p})^2}{2} + \dots$

4. If $\hat{A}$ and $\hat{B}$ are operators, prove

(a) that $\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$

(b) that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$

(c) that $\hat{A} + \hat{A}^\dagger, i\left(\hat{A} - \hat{A}^\dagger\right)$, and that $\hat{A}\hat{A}^\dagger$ are Hermitian operators.

Sol^n (a) Let $\hat{A}^* = \hat{C}$

given → $\int (\hat{A}\varphi_1)^* \cdot \varphi_2 = \int \varphi_1^* (\hat{C}\cdot\varphi_2)$ $\forall\, \varphi_1, \varphi_2$

To prove $\int (\hat{C}\varphi_1)^* \varphi_2 = \int \varphi_1^* (\hat{A}\cdot\varphi_2)$

Let us assume $\int (\hat{A}\varphi_2)^* \varphi_1 = \int \varphi_2^* (\hat{C}\varphi_1)$ For some $\hat{E}$

Also we have $\langle \varphi_1, \varphi_2 \rangle = \langle \varphi_2, \varphi_1 \rangle$ $\forall\, \varphi_1, \varphi_2$ (Inner product is commutative)

$$\int \varphi_1^* \varphi_2 = \int \varphi_2^* \varphi_1$$

$$\int \varphi_1^* \varphi_2 = \int \varphi_2 \, \varphi_1$$

$$\Rightarrow \int \varphi_1^* (\hat{A}\varphi_2) = \int (\hat{C}\varphi_1)^* \varphi_2$$

Hence Proved

(b) DIY

Hint: $\int (\hat{A}\hat{B}\varphi_1)^* \varphi_2 = \int (\hat{B}\varphi_1)^* (\hat{A})^* \varphi_2$ → Proceed from here

(c) Consider $\int (\hat{A}\hat{A}^* \underbrace{\varphi_1}_{\varphi_1'})^* \varphi_2 = \int (\hat{A}^* \varphi_1)^* \hat{A}^* \varphi_2 = \int \varphi_1 \underbrace{(\hat{A}^*)^*}_{\text{From (a)}} \hat{A}^* \varphi_2$

$$= \int \varphi_1 \hat{A}\hat{A}^* \varphi_2$$

Hence proved $(\hat{A}\hat{A}^*)^* = \hat{A}A^*$

Conclude

5. An operator is given by

$$\hat{G} = i\hbar\frac{\partial}{\partial x} + Bx$$

where B is a constant. Find the eigen function $\phi(x)$. If this eigen function is subjected to a boundary condition $\phi(a) = \phi(-a)$ find out the eigen values.

Sol^n $\hat{G} = i\hbar \frac{\partial}{\partial x} + Bx$

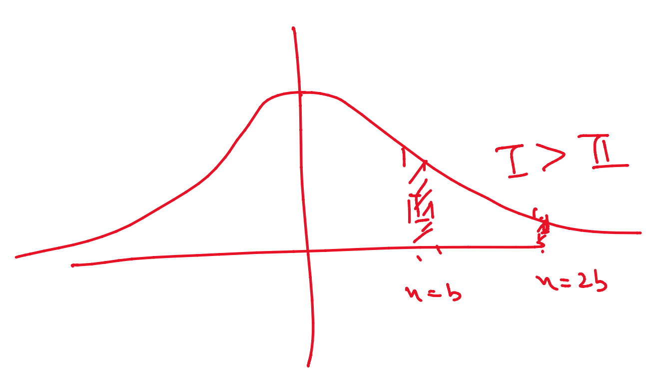
$\hat{G}\,\varphi(x) = c\,\varphi(x)$ $c \in \mathbb{C}$

$$i\hbar \frac{\partial \varphi(x)}{\partial x} + Bx\varphi(x) = c\,\varphi(x)$$

$$i\hbar \frac{d\varphi(x)}{\varphi(x)} = (c - Bx)\,dx \Rightarrow i\hbar \ln \varphi(x) = cx + D - \frac{Bx^2}{2}$$

↓

Solve using limits given



7. * Consider a large number (N) of identical experimental set-ups. In each of these, a single particle is described by a wave function $\Phi(x) = A\exp(-x^2/b^2)$ at $t = 0$, where A is the normalization constant and b is another constant with the dimension of length. If a measurement of the position of the particle is carried out at time $t = 0$ in all these set-ups, it is found that in 100 of these, the particle is found within an infinitesimal interval of $x = 2b$ to $2b + dx$. Find out, in how many of the measurements, the particle would have been found in the infinitesimal interval of $x = b$ to $b + dx$.

Sol^n $dP_1 = \Phi^*(2b)\cdot\Phi(2b)\cdot dx\Big|_{x=2b} = A\cdot A^*\, e^{-4}\cdot e^{-4}\cdot dx = \frac{100}{N}$

$dP_2 = \Phi^*(b)\cdot\Phi(b)\cdot dx = A\cdot A^*\, e^{-1}\cdot e^{-1} dx = \frac{100}{N}\,\frac{e^{-2}}{e^{-8}} = \frac{100}{N} e^{6} = \frac{N'}{N}$

$N' = 100\,e^{6} \simeq 40300$

8. * An observable A is represented by the operator $\hat{A}$. Two of its normalized eigen functions are given as $\Phi_1(x)$ and $\Phi_2(x)$, corresponding to distinct eigenvalues a_1 and a_2, respectively. Another observable B is represented by an operator $\hat{B}$. Two normalized eigen functions of this operator are given as $u_1(x)$ and $u_2(x)$ with distinct eigenvalues b_1 and b_2, respectively. The eigen functions $\Phi_1(x)$ and $\Phi_2(x)$ are related to $u_1(x)$ and $u_2(x)$ as, $\Phi_1 = D(3u_1+4u_2)$; $\Phi_2 = F(4u_1 - Pu_2)$ At time t = 0, a particle is in a state given by $\frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$. $= \psi_p$

$= \frac{1}{5}(4u_1 - 3u_2)$

(a) Find the values of D, F and P.

(b) If a measurement of A is carried out at t = 0, what are the possible results and what are their probabilities ?

(c) Assume that the measurement of A mentioned above yielded a value a1. If a measurement of B is carried out immediately after this, what would be the possible outcomes and what would be their probabilities ?

(d) If instead of following the above path, a measurement of B was carried out initially at t = 0, what would be the possible outcomes and what would be their probabilities ?

(e) Assume that after performing the measurements described in (c), the outcome was b_2. What would be the possible outcomes, if A were measured immediately after this and what would be the probabilities ?

(Explains the philosophy of QM very nicely)

(Stern-Gerlach Experiment)

$\hat{A}\,\Phi_1(x) = a_1\Phi_1(x)$

$\Phi_1(x) \rightarrow$ eigen functions

$\hat{B}\,u_1(x) = b_1 u_1(x)$

Sol^n (a) Φ_1 is normalised.

$\Rightarrow \langle \Phi_1, \Phi_1 \rangle = \int_{-\infty}^{\infty} \Phi_1^*(x)\cdot\Phi_1(x)\,dx = 1$

inner product

$= \langle D(3u_1+4u_2), D(3u_1+4u_2)\rangle \Rightarrow \int |D|^2 (3u_1^*+4u_2^*)(3u_1+4u_2)\,dx = 1$

$\Rightarrow |D|^2 \int 9u_1^*u + 16u_2^*u_2 + 12(u_1^*u_2+u_1u_2^*)\,dx = 1$

$\int \phi_1^*\phi = 1$ $\quad \int u_1^* u_1 = 1$

$\int \Phi_2^*\Phi_2 = 1$ $\quad \int u_2^* u_2 = 1$

$\int u_1^* u_1\,dx = 1$ & $\int u_2^* u_2\,dx = 1$

$\int_{-\infty}^{\infty} u_1^* u_2\,dx = 0 = \int_{-\infty}^{\infty} u_2^* u_1\,dx$ (Why ??) $u_1^* u_2 (x=0)$

Eigen functions corresponding to distinct eigen values of a Hermitian operator are orthogonal

$\Rightarrow |D|^2(25) = 1$

$\Rightarrow |D| = \frac{1}{5}$ (Assuming D to be real +ve) $\boxed{D = \frac{1}{5}}$

→ Assuming F & P to be real

$\langle \Phi_2, \Phi_2 \rangle = 1 = F^2(4^2+p^2)$

$\langle \Phi_2, \Phi_1 \rangle = 0 = \langle F(4u_1 - Pu_2), \frac{1}{5}(3u_1+4u_2)\rangle = \int_{-\infty}^{\infty} F(4u_1^* - Pu_2^*)\cdot\frac{1}{5}(3u_1+4u_2)\,dx$

$= \frac{12F}{5} - \frac{4PF}{5}$

Assume

$\hat{A} \rightarrow$ spin in z dir^n $\begin{cases}\Phi_1 \rightarrow +\frac{1}{2}\text{spin } \frac{\hbar}{2}\hat{z} \\ \Phi_2 \rightarrow -\frac{1}{2}\text{ spin } -\frac{\hbar}{2}\hat{z}\end{cases}$ $\Rightarrow$ $\boxed{P=3}$ & $\boxed{F = 1/5}$ (F = 0 is not possible)

$\hat{B} \rightarrow$ spin in x dir^n $\begin{cases}u_1 \\ u_2\end{cases}$

eigen functions of $\hat{A}$

(b) $\psi_p = \frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$

$t<0$ $\quad \psi_p = \frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$

$t>0$ $\quad \psi_p = \Phi_1$ or $\psi_p = \Phi_2$

$\frac{(2/3)^2}{(2/3)^2+(1/3)^2}$ $\quad \frac{(1/3)^2}{(1/3)^2+(2/3)^2}$

$\hat{A}\psi_p = \frac{2}{3}\hat{A}\Phi_1 + \frac{1}{3}\hat{A}\Phi_2 = \cancel{\frac{2}{3}a_1\Phi_1 + \frac{1}{3}a_2\Phi_2}$

Wavefunction collapse either into Φ_1 or Φ_2

→ Wavefunction collapse either into Φ_1 or Φ_2

(with probability $\left(\frac{(2/3)^2}{(2/3)^2+(1/3)^2} = \frac{4}{5}\right)$) (with probability $\left(\frac{(1/3)^2}{(1/3)^2+(2/3)^2} = \frac{1}{5}\right)$)

$$\psi_p = c_1\Phi_1 + c_2\Phi_2 + \cdots c_n\Phi_n$$

$$P(\Phi_1) = \frac{|c_1|^2}{|c_1|^2+|c_2|^2+\cdots |c_n|^2}$$

If measurement is carried out, value a_1 is obtained w.p. $(4/5)$

a_2 is obtained w.p. $(1/5)$

(c) Given that value a_1 is obtained.

$\Rightarrow$ wavefunction collapse into Φ_1

So new $\psi_p = \Phi_1$

eigenfunctions of $\hat{B}$

Now $\hat{B}\psi_p = \hat{B}\Phi_1 = \hat{B}\left(\frac{1}{5}(3u_1+4u_2)\right)$

Collapse into u_1 w.p. $\frac{(3/5)^2}{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2} = \frac{9}{25} \rightarrow b_1$

— — u_2 — — $\frac{16}{25} \rightarrow b_2$

(d) We always try to express the particle wave function of particle in terms of eigenfunctions of operator that is being acted upon it.

$$\psi_p = \frac{2}{3}\underset{\text{e.f. of }\hat{A}}{\Phi_1} + \frac{1}{3}\underset{\text{e.f. of }\hat{A}}{\Phi_2}$$

$$\psi_p = \frac{2}{3}\left(\frac{1}{5}(3u_1+4u_2)\right) + \frac{1}{3}\left(\frac{1}{5}(4u_1-3u_2)\right) = \frac{10}{15}u_1 + \frac{5}{15}u_2$$

$$= \frac{2}{3}u_1 + \frac{1}{3}u_2$$

$$\hat{B}\psi_p = \hat{B}\left(\frac{2}{3}u_1 + \frac{1}{3}u_2\right)$$

b_1 obtained w.p. $\frac{(2/3)^2}{\cdots} = \frac{4}{\cdots}$

b_2 obtained w.p. $\frac{(1/3)^2}{(1/3)^2+(2/3)^2} = \frac{1}{5}$ → different from ans in part c

b_1 obtained

w.p. $\dfrac{(2/3)^2}{(2/3)^2 + (1/3)^2} = \dfrac{4}{5}$

w.p. $\dfrac{(1/3)}{(1/3)^2 + (2/3)^2} = \dfrac{1}{5}$ $\longrightarrow$ different ...

Part c

(e) In (c) the particle collapses to u_2 ↳ e.f. of $\hat{B}$

$\Rightarrow$ $\psi_p = u_2$

$\Rightarrow$ $\hat{A}\psi_p = \hat{A}u_2$ ↳ Need to express this interms of e.f. of $\hat{A}$ (i.e. ϕ_1 & ϕ_2)

$$\phi_1 = \frac{3}{5}u_1 + \frac{4}{5}u_2$$

$$\phi_2 = \frac{4}{5}u_1 - \frac{3}{5}u_2$$

$$4\phi_1 - 3\phi_2 = u_2$$

$$\hat{A}(4\phi_1 - 3\phi_2)$$

a_1 measured

w.p. $\dfrac{4^2}{4^2+3^2} = \dfrac{16}{25}$

a_2 measured

w.p. $\dfrac{3^2}{3^2+4^2} = \dfrac{9}{25}$ $\rightarrow$ different froms ans in part (b)