Tutorial 10 Solutions

26 February 2022 14:22

Physical properties are same in all direction

5. * A two-dimensional isotropic harmonic oscillator has the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}k(x^2+y^2)$$

(a) Show that the energy levels are given by

$$E_{n_x,n_y} = \hbar\omega(n_x+n_y+1) \quad \text{where} \quad n_x, n_y \in (0,1,2...) \quad \omega = \sqrt{\frac{k}{m}}$$

(b) What is the degeneracy of each level?

$V(x) = \frac{1}{2}m\omega^2 x^2 + V_0$

$V(x,y,z) = V(x) + V(y) + V(z)$

$\Updownarrow$

$\varphi(x,y,z) = X(x)\cdot Y(y)\cdot Z(z)$

A separable wave function

Sol^n (a) TISE

$$\hat{H}\varphi(x,y) = E\varphi(x,y)$$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\cdot\varphi(x,y) + \frac{1}{2}k(x^2+y^2)\varphi(x,y) = E\varphi(x,y)$$

$-\frac{\partial^2 V}{\partial x^2} = \frac{\rho}{\varepsilon_0}$

CH 107

Ph 108

MA108

MA207

Expressing $\varphi(x,y) = X(x)\cdot Y(y)$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]X(x)Y(y) + \frac{1}{2}k(x^2+y^2)X(x)Y(y) = EX(x)Y(y)$$

Dividing by $X(x)\cdot Y(y)$

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2}\cdot\frac{1}{X(x)} + \frac{1}{2}kx^2\right]}_{E_x} + \underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2 Y(y)}{\partial y^2}\cdot\frac{1}{Y(y)} + \frac{1}{2}ky^2\right]}_{E_y} = E$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{2}kx^2 X(x) = E_x X(x)$$

$(k = m\omega^2)$

$\downarrow$

Solutions

$\downarrow$

$Ae^{-\alpha x^2}$

$$X_n(x) = \frac{1}{\sqrt{2^n\cdot n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)$$

$$E_{x_n} = \hbar\omega(n_x + 1/2)$$

$$E_{y_n} = \hbar\omega(n_y + 1/2)$$

$$\boxed{E_{Total} = \hbar\omega(n_x + n_y + 1)}$$

degeneracy $\longrightarrow$ (No. of microstates a particular macrostate has)

0

2

3

P, V, T, E $\rightarrow$ Macrostate

$E_0 = E_{00} = \hbar\omega$ (n_x n_y)

$\rightarrow E_1 = E_{01} = E_{10} = 2\hbar\omega$

$E \quad E \quad E \quad = E_{11} = 3\hbar\omega$

(x_1, p_1)

$\rightarrow E_1 = E_{01} = E_{10} = 2\hbar\omega$ 3

$E_2 = E_{20} = E_{02} = E_{11} = 3\hbar\omega$

$\vdots$

$E_n = E_{n0} - - - E_{0n} = (n+1)\hbar\omega$

① ✓ (n_1, P_1)

$\theta \rightarrow p$ (n_2, P_2)

$(n+1)$ $-p \leftarrow \theta$ ② ✓ $\vdots$

$-p \leftarrow \theta$

$\theta \rightarrow p$ (n_N, P_N)

6. * Consider the Hamiltonian of a two-dimensional anisotropic harmonic oscillator ($\omega_1 \neq \omega_2$)

$$H = \frac{p_1{}^2}{2m} + \frac{p_2{}^2}{2m} + \frac{1}{2}m\omega_1{}^2 q_1{}^2 + \frac{1}{2}m\omega_2{}^2 q_2{}^2$$

(a) Exploit the fact that the Schrödinger eigenvalue equation can be solved by separating the variables and find a complete set of eigenfunctions of H and the corresponding eigenvalues.

(b) Assume that $\frac{\omega_1}{\omega_2} = \frac{3}{4}$ .Find the first two degenerate energy levels. What can one say about the degeneracy of energy levels when the ratio between ω_1 and ω_2 is not a rational number.

$q_1, q_2 \rightarrow$ generalized coordinates

can be x, y

y, z

x, z

Soln (a) $\varphi(q_1, q_2) = \Phi_1(q_1)\,\Phi_2(q_2)$

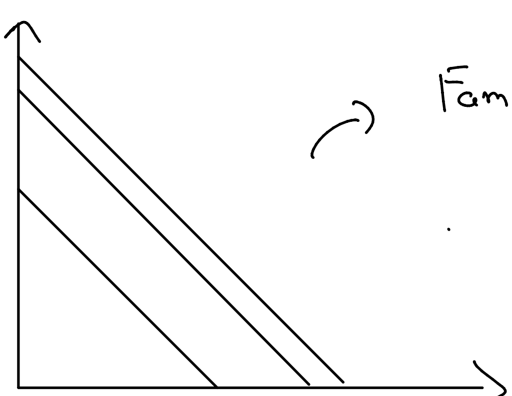
$$E(n_1, n_2) = \hbar\omega_1\left(n_1 + \frac{1}{2}\right) + \hbar\omega_2\left(n_2 + \frac{1}{2}\right)$$

(b) $\omega_1 = 3\omega$ $\omega_2 = 4\omega$ $n_1, n_2 \in \mathbb{Z}$

$$E(n_1, n_2) = \hbar\omega\left(3n_1 + 4n_2 + \frac{7}{2}\right) = \text{Const.}$$

$\frac{\omega_1}{\omega_2} = \frac{P}{q}$ $P, q \in \mathbb{Z}$ $q \neq 0$

n_1



n_2

Family of lines $\rightarrow$ $3n_1 + 4n_2 = C$

$n_1, n_2 \in \mathbb{Z} \Rightarrow C \in \mathbb{Z}$

First two degenerate levels

$(n_1, n_2) = (0, 3), (4, 0)$

$\frac{31}{2}\hbar\omega$

Take $\frac{\omega_1}{\omega_2} = P$; $P \in I$

$$E(n_1, n_2) = \hbar\omega_2\left(Pn_1 + n_2 + \frac{P+1}{2}\right)$$

For degenerate state $Pn_1 + n_2 = C$ for at least two pairs (n_1, n_2)

$Pn_1 + n_2 = Pn_1' + n_2'$ $n_1' \neq n_1$

$\underbrace{P(n_1 - n_1')}_{\in I} = \underbrace{(n_2' - n_2)}_{\in Q}$ $n_2' \neq n_2$

(n_2', n_1') exist

$\hookrightarrow \in \mathbb{Q}$

$\in I$

so no two distinct pairs (n_1, n_2) & (n_1', n_2') exist in this case.

Hence degeneracy is not possible

7. A particle of mass m is confined to move in the potential $(m\omega^2 x^2)/2$. Its normalized wave function is

$$\psi(x) = \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-(\beta x^2/2)}$$

where β is a constant of appropriate dimension.

(a) Obtain a dimensional expression for β in terms of m, ω and $\hbar$.

(b) It can be shown that the above wave function is the linear combination

$$\psi(x) = a\psi_0(x) + b\psi_2(x)$$

where $\psi_0(x)$ is the normalized ground state wave function and $\psi_2(x)$ is the normalized second excited state wave function of the potential. Evaluate b and hence calculate the expectation value of the energy of the particle in this state $\psi(x)$.

Given: $I_0(\beta) = \int_{-\infty}^{+\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$, $\quad I_n(\beta) = \int_{-\infty}^{+\infty} (x^2)^n e^{-\beta x^2} dx = (-1)^n \frac{\partial^n}{\partial \beta^n}(I_0(\beta))$,

$\psi_0(x) = \left(\frac{\beta}{\pi}\right)^{1/4} e^{\frac{-\beta x^2}{2}}$

Solⁿ $\quad [\beta x^2] = M^0 L^0 T^0$

$[\beta] = L^{-2} = [M]^a [\omega]^b [\hbar]^c = [M]^a [T^{-1}]^b [M L^2 T^{-1}]^c$

$c = -a$

$\beta = \dfrac{m\omega}{\hbar} \qquad c = -b$

$c = -1$

(b) $\quad \varphi(x) = a\,\varphi_0(x) + b\,\varphi_2(x)$

$\int_{-\infty}^{\infty} \varphi(x)\cdot \varphi_2^*(x)\,dx = \cancel{\int_{-\infty}^{\infty} a\,\varphi_0(x)\cdot\varphi_2(x)\,dx}^{\,0} + b\int_{-\infty}^{\infty} \underbrace{\varphi_2(x)\cdot\varphi_2^*(x)}_{1}\,dx$

(orthogonal wave functions) $\quad 1 \rightarrow$ Normalized wavefunction

$b = \int_{-\infty}^{\infty} \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-\beta x^2/2} \cdot \frac{1}{2\sqrt{2}}\left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2} H_2(\sqrt{\beta}\,x)$

$\varphi_n(x) = \dfrac{1}{2}$

$H_2(z) = e^{-z^2}\dfrac{d^2}{dz^2}\left(e^{-z^2}\right) = e^{-z^2}\left(\dfrac{d}{dz}\left(-2z\,e^{-z^2}\right)\right)$

$= 4z^2 e^{-2z^2} - 2e^{-2z^2}$

$b = \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/2} \cdot \frac{1}{2\sqrt{2}} \int_{-\infty}^{\infty} x^2 e^{-\beta x^2}\left[4\beta x^2 e^{-2\beta x^2} - 2e^{-2\beta x^2}\right] \rightarrow 2\sqrt{\frac{\pi}{\beta^3}}$

$\int^{\infty} 4\beta x^4 e^{-3\beta x^2}\,dx - \int 2x^2 e^{-3\beta x^2}\,dx$

$\times \sqrt{\pi}$

$$\int_{-\infty} 4\beta x^4 e^{-3\beta x^2}\,dx - \int 2x^2 e^{\cdots}\,dx$$

$$b = \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/2}\cdot\frac{1}{\cancel{2}\sqrt{2}}\cdot\cancel{2}\sqrt{\frac{\pi}{\beta^3}}$$

$I_2(3\beta)$

$= (-1)^2\cdot\frac{3}{4}\sqrt{\frac{\pi}{\beta^5}}$

$I_1(3\beta)$

$= (-1)^1\left(-\frac{1}{2}\right)\sqrt{\frac{\pi}{\beta^3}}$

$$\boxed{b = \sqrt{\frac{2}{3}}}$$

$\varphi(x) = a\varphi_0 + b\varphi_2$

$\varphi^*(x) = a\varphi_0^* + b\varphi_2^*$

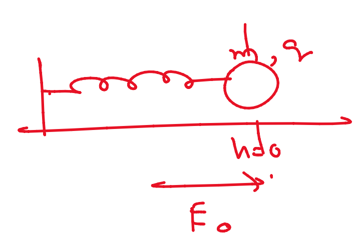
$a^2 + b^2 = 1$

$$\boxed{a = \frac{1}{\sqrt{3}}}$$

$$\int \varphi(x)\cdot\varphi^*(x) = \int a^2\varphi_0\varphi_0^* + b^2\int\varphi_2\varphi_2^* + ab\int \cancel{\varphi_0\varphi_2^*} + ab\int\cancel{\varphi_0^*\varphi_2}$$

$1 = a^2 + b^2$

9. * A charged particle of mass ' m ' and charge ' q ' is bound in a 1-dimensional simple harmonic oscillator potential of angular frequency ' ω '. An electric field E_0 is turned on.
(a) What is the total potential V(x) experienced by the charge ?
(b) Express the total potential in the form of an effective harmonic oscillator potential.
(c) Sketch V (x) versus x.
(d) What is the ground state energy of the particle in this potential?
(e) What is the expectation value of the position (x) if the charge is in its ground state ?



$x = \frac{qE_0}{m}$

$-\frac{qE_0}{m}$

$E = -\frac{dV}{dx}$

$V = -E\cdot dx$

Sol^n (a) $V(x) = -qE_0x + \frac{1}{2}m\omega^2x^2$; The $\vec{E} = E_0$ is assumed to be in $+x$ dir^n.

$\frac{1}{2}m\omega^2x^2 + V_0$

(b) $V(x) = \frac{1}{2}m\omega^2x^2 - qE_0x$

$$= \frac{1}{2}m\omega^2\left(x^2 - \frac{2qE_0}{m\omega^2}x\right)$$

$$\frac{1}{2}m\omega^2\left(x^2 - \frac{2qE_0}{m\omega^2}x\right)$$

$$\left(x - \frac{qE_0}{m\omega^2}\right)^2$$

$$= \frac{1}{2}m\omega^2\left(x - \frac{qE_0}{m\omega^2}\right)^2 - \frac{1}{2}m\cancel{\omega^2}\left(\frac{q^2E_0^2}{m^2\omega^{4}}\right)$$

$$V(x) = \frac{m\omega^2}{2}\left(x - \frac{qE_0}{m\omega^2}\right)^2 - \frac{q^2E_0^2}{2m\omega^2}$$

$$= \frac{1}{2}m\omega^2x'^2 + V_0 \rightarrow x' = x - \frac{qE_0}{m\omega^2} \rightarrow \text{mean pos}^n \text{ shifted}$$

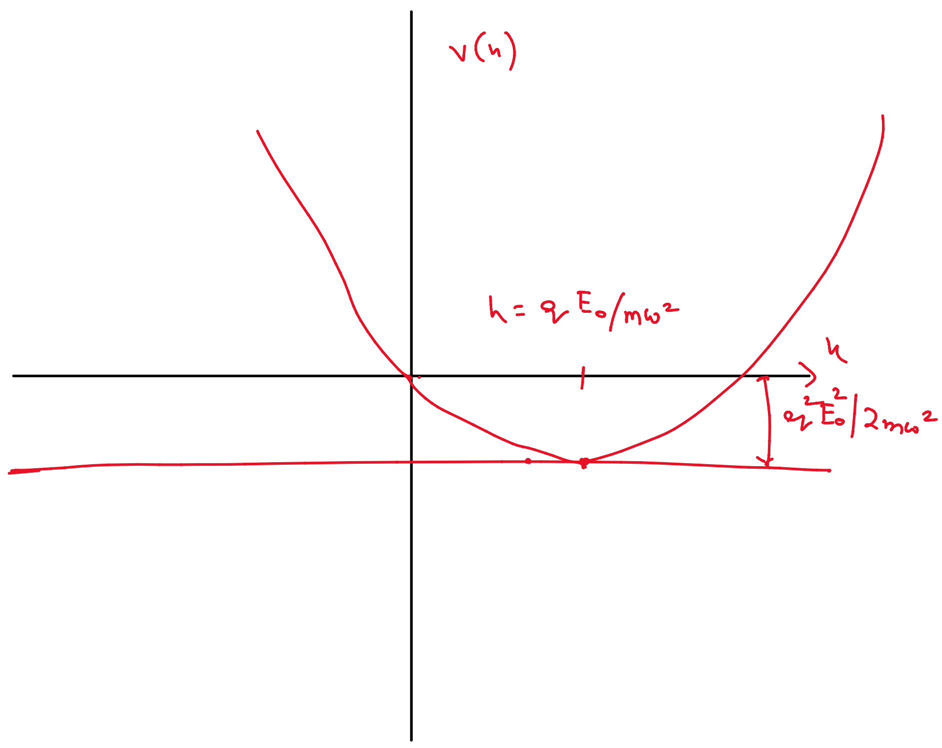
$$= \frac{1}{2} m\omega^2 x'^2 + V_0 \longrightarrow x' = x_0 - \frac{qE_0}{m\omega^2} \longrightarrow \text{mean shifted}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \varphi}{\partial x^2} + \left(\frac{1}{2} m\omega^2 \underbrace{\left(x - \frac{qE_0}{m\omega^2}\right)}^2 + V_0\right)\varphi(x) \longrightarrow V_0 = -\frac{q^2 E_0^2}{2m\omega^2}$$

$$= E\varphi(x)$$

$$x - \frac{qE_0}{m\omega^2} = x'$$

(c)

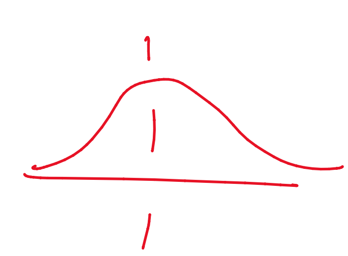


(d) $E = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{q^2 E_0^2}{2m\omega^2}$

(e) Ground state wave function

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-(m\omega/2\hbar)\underline{x'^2}} \qquad \left\{ x' = x - \frac{qE_0}{m\omega^2} \right\}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \varphi_0(x) \cdot x \cdot \varphi_0^*(x) \cdot dx$$



$$x' \rightarrow x - \frac{qE_0}{m\omega^2}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \cancel{\varphi_0(x)\, x' \cdot \varphi_0^*(x) \cdot dx'}^{\,0} + \underbrace{\int_{-\infty}^{\infty} \varphi_0(x) \cdot \varphi_0^*(x) \cdot dx'}_{1} \left(\frac{qE_0}{m\omega^2}\right)$$

$$= \frac{qE_0}{m\omega^2}$$

$$= \frac{q\,[illegible]}{m\omega^2}$$

① $\Delta x = \langle x^2\rangle - \cancel{\langle x\rangle^2}^{\to 0}$ $\qquad V(x) = \frac{1}{2}m\omega^2 x^2$

$$= \langle x^2\rangle$$

$$= \int \varphi_0(x)\cdot x^2\cdot \varphi_0^*(x)\,dx$$

$$= \int \left(\frac{\beta}{\pi}\right)^{1/4} \underbrace{e^{-\frac{\beta x^2}{2}}\cdot x^2\cdot e^{-\beta x^2/2}}\cdot\left(\frac{\beta}{\pi}\right)^{1/4} dx \qquad \beta = \frac{m\omega}{\hbar}$$

Δx

$\Delta p = \langle p^2\rangle - \langle p\rangle^2$

$E \sim \frac{(\Delta p)^2}{2m}$

$$\int \varphi_0^*(x)\, -i\hbar\frac{\partial}{\partial x}\,\varphi_0(x)$$

$$\int Ae^{-\alpha x^2}\; -i\hbar\left(-\alpha A e^{-\alpha x^2}\cdot 2x\right)$$

$$\int_{-\infty}^{\infty} \underbrace{(\;)\, e^{-2\alpha x^2}\cdot x}$$

⑧ $n_x + n_y + n_z = n$ $\qquad g_n = \frac{1}{2}(n+1)(n+2)$