Tutorial 3 Solution

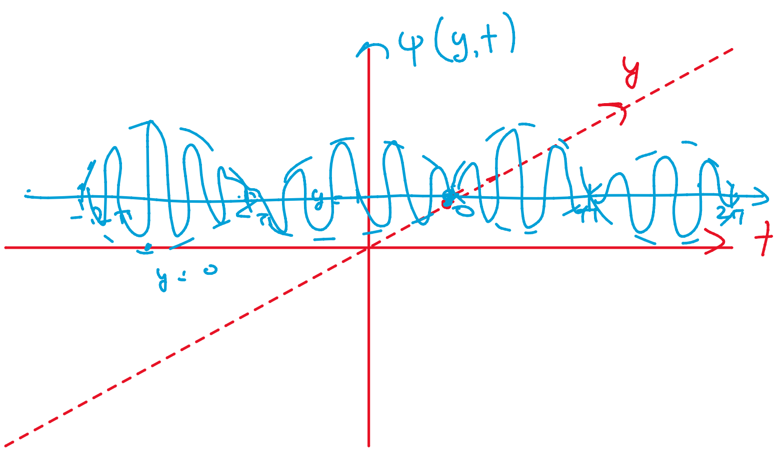
26 December 2021 13:31

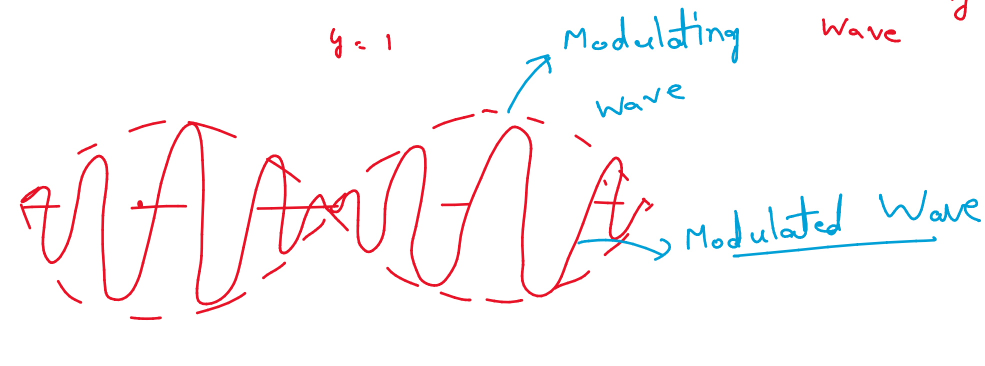
1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

Sol^n $\varphi(y,t) = \varphi_1(y,t) + \varphi_2(y,t) = 5y\cos 7t - 5y\cos 9t = 5y\,(2\sin t\,\sin 8t) = 10y\sin t\,\sin 8t$

$10y\sin t$ → Modulating Wave

$\sin 8t$ → Modulated Wave





2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(\underset{k_1}{8.0x} - 400t) \quad \& \quad y_2 = 0.002\cos(\underset{k_2}{7.6x} - 380t)$$

where x, y are in meters and t is in sec.

(a) Find the resultant wave and its phase and group velocities

(b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ? [Ans.: $v_p = 50$ m/s, $v_g = 50$ m/s, $\Delta x = 5\pi$ m, $\Delta x \Delta k = 2\pi$]

Sol^n (a) $y = y_1 + y_2$

$$= 0.002\left[2\cos(7.8x - 390t)\cos(0.2x - 10t)\right]$$

$$= 0.004\,\underbrace{\cos(7.8x - 390t)}_{\text{high freq. wave packet}}\,\underbrace{\cos(0.2x - 10t)}_{\text{low freq. envelope}}$$

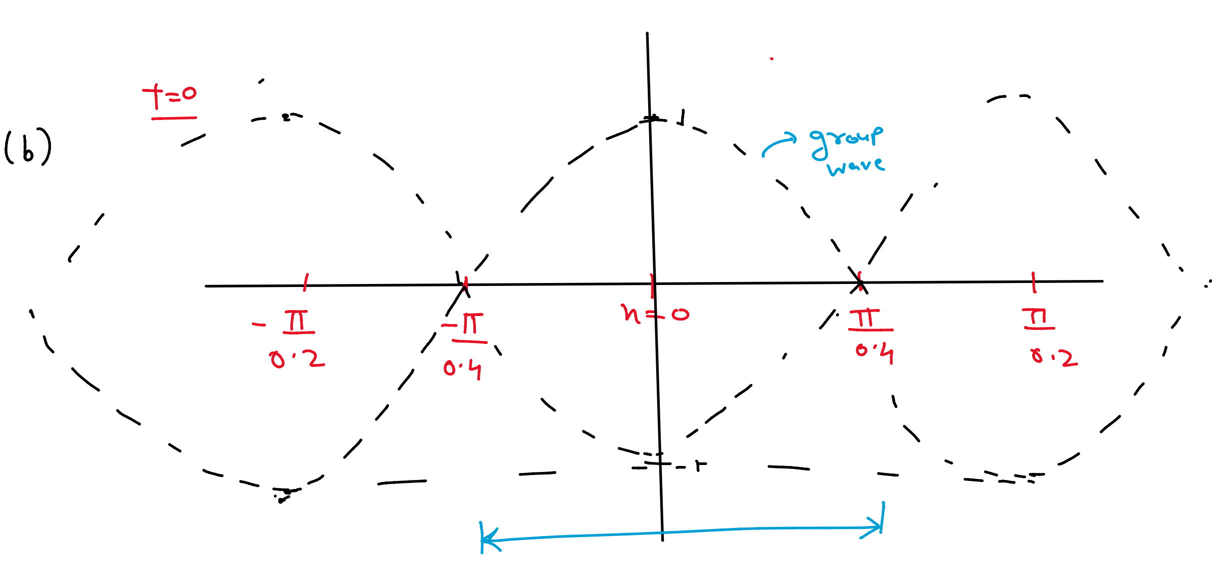
→ $t = 0$ → $\cos(0.2x)$

V_g → velocity of envelope $= \dfrac{\omega_e}{k_e} = \dfrac{10}{0.2} = 50$ m/s

V_p → velocity of wave packet $= \dfrac{\omega_p}{k_p} = \dfrac{390}{7.8} = 50$ m/s

→ Coincidence ??

(b)



$$\Delta x = \frac{2\pi}{0.4} = 5\pi$$

$$\Delta k \rightarrow |k_2 - k_1| = 0.4$$

$$\boxed{\Delta x \cdot \Delta k = 2\pi}$$

$$p = \hbar k$$

$$\Delta p = \hbar \Delta k$$

$$\Delta x \cdot \frac{\Delta p}{\hbar} = 2\pi$$

$$\Delta x \cdot \Delta p = 2\pi\hbar > \hbar/2$$

(4)

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

Sol^n

$$E^2 = p^2c^2 + m_0^2c^4 \qquad p = \hbar k \qquad E = \hbar\omega$$

$$\hbar^2\omega^2 = \hbar^2k^2c^2 + m_0^2c^4$$

$$\omega^2 = k^2c^2 + \frac{m_0^2c^4}{\hbar^2} \rightarrow \text{Dispersion Relation}$$

$$\omega = \sqrt{k^2c^2 + \frac{m_0^2c^4}{\hbar^2}}$$

$$\frac{d\omega}{dk} = \frac{kc^2}{\sqrt{k^2 + \frac{m_0^2c^4}{\hbar^2}}}$$

$$V_p = \frac{\omega}{k} = \frac{\sqrt{k^2c^2 + \frac{m_0^2c^4}{\hbar^2}}}{k} = \sqrt{c^2 + \frac{m_0^2c^4}{k^2\hbar^2}} = \frac{mc^2}{mv} = \frac{c^2}{v} > c$$

$$V_g = \frac{d\omega}{dk} = \frac{kc^2}{\sqrt{k^2c^2 + \frac{m_0^2c^4}{\hbar^2}}} = v \rightarrow \text{velocity of particle}$$

(5)

$Ae^{i(kx-\omega t)}$

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

(a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ε and permeability μ.

(b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

Sol^n (a) $V = \frac{1}{\sqrt{\varepsilon\mu}}$

Dispersion Relation

↑

$\sqrt{\varepsilon\mu}$

(b) $\varepsilon = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$

$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)\mu_0}} \Rightarrow \frac{\cancel{\omega}}{k} = \frac{c\cancel{\omega}}{\sqrt{\omega^2-\omega_p^2}} \Rightarrow \boxed{k = \frac{\sqrt{\omega^2-\omega_p^2}}{c}}$ — ①

(c) differentiating ① w.r.t ω

$$\frac{dk}{d\omega} = \frac{\omega}{c\sqrt{\omega^2-\omega_p^2}} = \frac{1}{V_g}$$

for $\omega = 3\omega_p$

$k = \frac{2\sqrt{2}\,\omega_p}{c} \Rightarrow \boxed{V_p = \frac{\omega}{k} = \frac{3c}{2\sqrt{2}}} > c$

$$\boxed{V_g = \frac{2\sqrt{2}\,c}{3}}$$

$$\boxed{V_p \cdot V_g = c^2}$$

6

6. The dispersion relation for a lattice wave propagating in a 1-D chain of atoms of mass m bound together by a force constant β is given by $\omega = \omega_0 \sin\left(\frac{ka}{2}\right)$, where $\omega_0 = \sqrt{4\beta/m}$ and a is the distance between the atoms.

(a) Show that the medium is non- dispersive in the long wavelength limits.

(b) Find the group and phase velocities at $k = \pi/a$. [Ans.: 0, $\omega_o a/\pi$]

Dispersive $\rightarrow$ $V_g \neq V_p$

Non dispersive $\rightarrow$ $V_g = V_p$

Sol^n (a) $V_g = \frac{d\omega}{dk} = \frac{\omega_0 a}{2}\cos\left(\frac{ka}{2}\right)$

$V_p = \frac{\omega}{k} = \frac{\omega_0 \sin\left(\frac{ka}{2}\right)}{k}$

Non Dispersive medium $\rightarrow$ $V_g = V_p$

$\lim_{\lambda\to\infty} V_g = \lim_{k\to 0} V_g = \frac{\omega_0 a}{2}$

$\lim_{\lambda\to\infty} V_p = \lim_{k\to 0} V_p = \frac{\omega_0 a}{2}$

$\lim_{k\to 0} \frac{\omega_0 \sin\left(\frac{ka}{2}\right)}{ka/2}\left(\frac{a}{2}\right) \to \left(\frac{\omega_0 a}{2}\right)$

In longer wavelength limit $V_g = V_p$ holds

(b) $k = \pi/a \Rightarrow V_g = 0, \; V_p = \frac{\omega_0 a}{\pi}$

7

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x(1 - \cos k_x a_x) + 2\beta_y(1 - \cos k_y a_y) = 0 \quad \text{— ①}$$

where $\vec{k} = k_x\hat{i} + k_y\hat{j}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis

$V_g = \frac{d\omega}{dk} = \left(\frac{\partial\omega}{\partial k_x}\hat{i} + \frac{\partial\omega}{\partial k_y}\hat{j}\right) = \nabla_{\vec{k}}(\omega)$

the angle that it makes with the x-axis

Sol^n $V_{group} = \frac{d\omega}{d\vec{k}} = \left(\frac{\partial \omega}{\partial k_x}\hat{i} + \frac{\partial \omega}{\partial k_y}\hat{j}\right) = \nabla_{\vec{k}}(\omega)$

$$V_{group,x} = \frac{\partial \omega}{\partial k_x}$$

Diff^n eq^n ① w.r.t. x

$$-2m\omega \frac{\partial \omega}{\partial k_x} + 2\beta_x a_x \sin(k_x a_x) = 0$$

$$\frac{\partial \omega}{\partial k_x} = \frac{\beta_x a_x}{m\omega} \sin(k_x a_x) = V_{group,x}$$

Similarly $\frac{\partial \omega}{\partial k_y} = \frac{\beta_y a_y}{m\omega}(\sin k_y a_y) = V_{group,y}$

$$\cos\theta = \frac{V_{group,x}}{V_{group}} = \frac{\beta_x a_x \sin(k_x a_x)}{\sqrt{\beta_x^2 a_x^2 \sin^2(k_x a_x) + \beta_y^2 a_y^2 \sin^2(k_y a_y)}}$$