Tutorial 6 Solutions

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1. *Show that

$$\psi(x) = A\,\sin(kx) + B\,\cos(kx)$$

and

$$\psi(x) = Ce^{ikx} + De^{-ikx}$$

are equivalent solutions of TISE of a free particle. A, B, C and D can be complex numbers.

Sol^n

$$\psi(x) = A\sin kx + B\cos kx = A\left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) + B\left(\frac{e^{ikx} + e^{-ikx}}{2}\right)$$

$$= e^{ikx}\underbrace{\left(\frac{A}{2i} + \frac{B}{2}\right)}_{C} + e^{-ikx}\underbrace{\left(-\frac{A}{2i} + \frac{B}{2}\right)}_{D}$$

2. Show that

$$\Psi(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)$$

does not obey the time-dependant Schroedinger's equation for a free particle.

Sol^n TDSE

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} = i\hbar\frac{\partial \psi(x,t)}{\partial t}$$

$$+\frac{\hbar^2 k^2}{2m}\psi(x,t) = -i\hbar\omega A\cos(kx-\omega t) + i\hbar\omega B\sin(kx-\omega t)$$

Clearly $\psi(x,t)$ is not an eigenfunction of $\left(-i\hbar\frac{\partial}{\partial t}\right)$. Hence it isn't a valid sol^n of TDSE.

4. * A free proton has a wave function given by

$$\Psi(x,t) = Ae^{i(5.02*10^{11}x - 8.00*10^{15}t)} = Ae^{i(kx-\omega t)}$$

The coefficient of x is inverse meters, and the coeffcient of t is inverse seconds. Find its momentum and energy.

Sol^n

$$\psi(x,t) = Ae^{i\alpha x}\cdot e^{-i\beta t} = \phi(x)\cdot T(t)$$

↳ eigen function of both $\hat{p}\left(-i\hbar\frac{\partial}{\partial x}\right)$ & $\hat{H}\left(i\hbar\frac{\partial}{\partial t}\right)$

$$\hat{p}\psi(x,t) = -i\hbar\frac{\partial}{\partial x}\left(Ae^{i\alpha x}\cdot e^{-i\beta t}\right) = \hbar\alpha\left(Ae^{i\alpha x}\cdot e^{-i\beta t}\right)$$

$$= \hbar\alpha\,\psi(x)$$

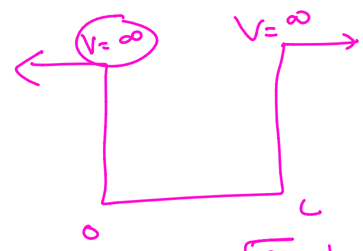
$$\boxed{P = \hbar\alpha} \qquad \boxed{E = \hbar\beta}$$

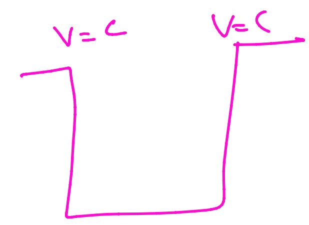
$$\psi(x,t) = \underline{Ae^{i(kx-\omega t)}}$$

5. A particle moving in one dimension is in a stationary state whose wave function,

$$\Psi(x)=\begin{cases}0, & x<-a\\ A\left(1+\cos\frac{\pi x}{a}\right), & -a\le x\le a\\ 0, & x>a\end{cases}$$

where A and a are real constants.
(a) Is this a physically acceptable wave function? Explain.
(b) Find the magnitude of A so that $\psi(x)$ is normalized.
(c) Evaluate Δx and Δp. Verify that $\Delta x\Delta p\ge \hbar/2$. $(\hbar/2)$
(d) Find the classically allowed region.





$[0, L]$

$\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

$A\cos\left(\frac{n\pi x}{L}\right)$ $\rightarrow x=0$ $\rightarrow A \rightarrow \varphi'(x=0^+)$; $0\rightarrow\varphi'(x=0^-)$

Sol^n (a) (i) $\varphi(-a^-)=0$ $\quad\varphi(-a^+)=0$

$\varphi(a^-)=0$ $\quad\varphi(a^+)=0$

(ii) $\varphi'(a^-)=0$ $\quad\varphi'(a^+)=0$

$\varphi'(-a^-)=0$ $\quad\varphi'(-a^+)=0$

(iii) Normalizability

$$\int_{-\infty}^{\infty}\varphi^*\cdot\varphi\,dx=\int_{-a}^{a}|A|^2\left(1+\cos\frac{\pi x}{a}\right)^2 dx$$

$$=\int_{-a}^{a}|A|^2\left[4\cos^4\frac{\pi x}{2a}\right] dx$$

$\frac{\pi x}{2a}=t$ $\qquad dx=\frac{2a\,dt}{\pi}$

$$=\frac{8a}{\pi}|A|^2\underbrace{\int_{-\pi/2}^{\pi/2}\cos^4 t\,dt}_{2\times 3\pi/16}$$

$$=3a|A|^2$$ ↪ finite

Hence $\varphi(x,t)$ is normalizable

(b) $3a|A|^2=1$

$$\boxed{A=\frac{1}{\sqrt{3a}}}$$

(c) $\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$

$$\langle x\rangle=\int_{-\infty}^{\infty}\varphi^*(x)\cdot x\cdot\varphi(x)\,dx=\int_{-a}^{a}\frac{4x}{3a}\cos^4\left(\frac{\pi x}{2a}\right)dx=0$$ (Why?)

$$\langle x^2\rangle=\int_{-\infty}^{\infty}\varphi^*(x)\cdot x^2\cdot\varphi(x)\,dx=\int_{-a}^{a}\frac{4x^2}{3a}\cos^4\left(\frac{\pi x}{2a}\right)dx=\int_{0}^{a}\frac{8x^2}{3a}\cos^4\left(\frac{\pi x}{2a}\right)dx$$

$$=\frac{8}{3a}\int_0^a x^2\left[1+\cos\frac{\pi x}{a}\right]^2 dx$$

$$= \frac{8}{3a}\int_0^a x^2 + 2x^2\cos\frac{\pi x}{a} + x^2\cos^2\frac{\pi x}{a}\,.dx$$

$$= \frac{8}{3a}\int_0^a \frac{3x^2}{2} + 2x^2\cos\frac{\pi x}{a} + \frac{x^2}{2}\cos\frac{2\pi x}{a}\,.dx$$

$$= \frac{8}{3a}\Bigg[\frac{a^3}{2} + \cancel{\frac{2x^2\sin\pi x/a}{\pi/a}}\Bigg|_0^a - \frac{4}{(\pi/a)}\int_0^a x\sin\left(\frac{\pi x}{a}\right) + \cancel{\frac{x^2\sin\left(\frac{2\pi x}{a}\right)}{2(2\pi/a)}}\Bigg|_0^a$$

$$- \frac{2}{2(2\pi/a)}\int_0^a x\sin\left(\frac{2\pi x}{a}\right)\Bigg]$$

$$= \frac{8}{3a}\Bigg[\frac{a^3}{2} - \frac{4}{(\pi/a)}\Bigg[\frac{x\cos(\pi x/a)}{\pi/a}\Bigg|_a^0 + \frac{\sin(\pi x/a)}{(\pi/a)^2}\Bigg|_0^a\Bigg]$$

$$- \frac{1}{(2\pi/a)}\Bigg[\frac{x\cos(2\pi x/a)}{2\pi/a}\Bigg|_a^0 + \frac{\sin(2\pi x/a)}{(2\pi/a)^2}\Bigg|_0^a\Bigg]\Bigg]$$

$$= \frac{8}{3a}\left[\frac{a^3}{2} - \frac{4}{(\pi/a)}\left[\frac{a^2}{\pi}\right] - \frac{1}{(2\pi/a)}\left[-\frac{a^2}{2\pi}\right]\right]$$

$$= \frac{8}{3a}\left[\frac{a^3}{2} - \frac{4a^3}{\pi^2} + \frac{a^3}{4\pi^2}\right] = \frac{8}{3a}\left[\frac{a^3}{2} - \frac{15a^3}{4\pi^2}\right] = \frac{8a^2}{3}\left[\frac{2\pi^2-15}{4\pi^2}\right]$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle} = \sqrt{\frac{2a^2}{3}\left[\frac{2\pi^2-15}{\pi^2}\right]}$$

$$\langle p\rangle = \int_{-a}^{a}\psi^*(x)\,\hat{p}\psi(x)\,.dx = \int_{-a}^{a} A^*\left(1+\cos\frac{\pi x}{a}\right) -i\hbar\frac{\partial}{\partial x}\left[A\left(1+\cos\frac{\pi x}{a}\right)\right].dx$$

$$= \int_{-a}^{a}\frac{i\hbar}{3a}\cdot\left(\frac{\pi}{a}\right)\left[\sin\left(\frac{\pi x}{a}\right) + \frac{\sin(2\pi x/a)}{2}\right].dx$$

$$= \frac{i\hbar\pi}{3a^2}\cdot 0 = 0$$

$$\langle p^2\rangle = \int^{a} A^*\left(1+\cos\frac{\pi x}{a}\right)\cdot -\hbar^2\frac{\partial^2}{\partial x^2}A\left(1+\cos\frac{\pi x}{a}\right)\cdot dx$$

$$\langle p^2\rangle = \int_{-a} A^*\left(1+\cos\frac{\pi x}{a}\right)\cdot -\hbar^2 \frac{\partial}{\partial x^2}\left(1+\cos\frac{\pi x}{a}\right)$$

$$= +\frac{\hbar^2}{3a}\left(\frac{\pi}{a}\right)^2 \int_{-a}^{a}\left(\cos\frac{\pi x}{a} + \underbrace{\cos^2\frac{\pi x}{a}}_{(1+\cos 2\pi x/a)/2}\right) dx$$

$$= \frac{\hbar^2}{3a}\left(\frac{\pi}{a}\right)^2\left[0+\frac{1}{2}\times 2a + 0\right] = \frac{\hbar^2\pi^2}{3a^2}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle}$$

(I was proud to be a TA until I solved this question)

$$\Delta x\cdot\Delta p = \sqrt{\frac{2\cancel{a}}{3}\left[\frac{2\pi^2-15}{\cancel{\pi^2}}\right]\cdot\frac{\hbar^2\cancel{\pi^2}}{3\cancel{a}}} = \sqrt{\frac{2\hbar^2}{9}\left[2\pi^2-15\right]} = \hbar\sqrt{\frac{(2\pi^2-15)}{9}} \simeq \hbar\sqrt{\frac{5}{9}} > \frac{\hbar}{2}$$

(d) Classically allowed region → $x \in (-a, a)$

↳ $\psi(x)$ is non zero

$\underbrace{\{V < E\}}$ Classically Allowed energy $\qquad \{V > E\}$

$\psi(x,t) = \phi(x)\cdot T(t)$ → TDSE $\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\cdot\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}\right)$

$\phi(x)\cdot e^{-iEt/\hbar}$

TISE

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} + V(x)\,\phi(x) = E\,\phi(x)$$

$$i\hbar\frac{\partial T(t)}{\partial t} = E\,T(t)$$

$$T(t) = e^{-iE/\hbar\, t}$$

$|\psi(x,t)|^2 =$

P.d. $= \psi(x,t)\cdot\psi^*(x,t)$

$= \left(\phi(x)\cdot \cancel{e^{-iE/\hbar t}}\right)\cdot\left(\phi^*(x)\cdot \cancel{e^{iE/\hbar t}}\right)$

$= \boxed{|\phi(x)|^2}$ → Stationary State

→ All separable solutions correspond to stationary states.

$$dP = \psi(x,t)\cdot\psi^*(x,t)\,dx = \phi(x)\cdot\phi^*(x)\left(e^{-i\omega t}\cdot e^{i\omega t}\right)\cdot dx = |\phi(x)|^2\cdot dx$$

Probability independent of time

(Stationary)

Q. Check if given wavefunctions are stationary wavefunctions

(i) $\psi_1(x,t) = C_1 e^{-i\omega_0 t} + 5C_3 e^{-i\omega_0 t} = (C_1+5C_3)e^{-i\omega_0 t}$

$C e^{-i\omega_0 t}$ → $\hbar\omega_0$

(ii) $\psi_2(x,t) = C_2 e^{-i(3\omega_0)t} + C_4 e^{-i(5\omega_0)t}$

$(3\hbar\omega_0$ or $5\hbar\omega_0)$ → wavefunction collapse

states

$(3\hbar\omega_0 \text{ or } 5\hbar\omega_0) \rightarrow$ Wavefunction ...

$i\hbar \frac{\partial}{\partial t}$

$\rightarrow$ Stationary states are definite energy states

6. * Consider the 1-dimensional wave function of a particle of mass m, given by

$n > 0$
$$\psi(x) = A\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}} \rightarrow \text{Hydrogen like atom}$$

where, A, n and x_0 are real constants.
(a)Find the potential $V(x)$ for which $\psi(x)$ is a stationary state (It is known that $V(x) \rightarrow 0$ as $x \rightarrow \infty$).
(b)What is the energy of the particle in the state $\psi(x)$?

Sol^n
$$\underbrace{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)}_{\text{must be an energy eigenfunction}} = E\psi(x)$$

$$-\frac{\hbar^2 A}{2m}\left[\frac{n x^{n-1}}{x_0^n}\cdot e^{-x/x_0} + \frac{x^n}{x_0^n} e^{-x/x_0}\cdot\left(-\frac{1}{x_0}\right)\right] + V(x)\cdot A\left(\frac{x}{x_0}\right)^n e^{-x/x_0} = E\psi(x)$$

$$-\frac{\hbar^2 A}{2m}\left[\frac{n(n-1)x^{n-2}}{x_0^n}e^{-x/x_0} - \frac{n x^{n-1}}{x_0^{n+1}}e^{-x/x_0} + \frac{n x^{n-1}}{x_0^n}e^{-x/x_0}\left(-\frac{1}{x_0}\right) - \frac{x^n}{x_0^{n+2}}e^{-x/x_0}\right]$$
$$+ V(x)\left(\quad\right)$$

$$\underbrace{A\left(\frac{x}{x_0}\right)^n e^{-x/x_0}}_{\psi(x)}\left[-\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x x_0} - \frac{1}{x_0^2}\right] + V(x)\right] = E\psi(x)$$

$$\Rightarrow \boxed{V(x) = \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x x_0}\right]}$$

$V(x) \rightarrow 0$ as $x \rightarrow \infty$)

$c = 0$

(b) $$\boxed{E = -\frac{\hbar^2}{2m x_0^2}}$$