Tutorial 7 Solutions

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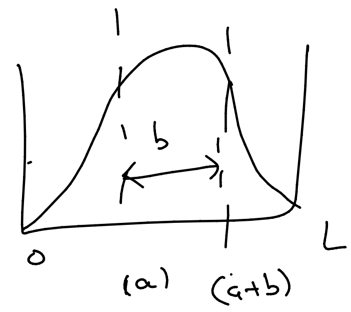
1. * For a particle in a 1-D box of side L, show that the probability of finding the particle between $x = a$ and $x = a + b$ approaches the classical value b/L, if the energy of the particle is very high.



b/L (classically)

Sol^n $\quad \psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ $\qquad E_n = \frac{h^2 n^2}{8mL^2} \uparrow \; n >> 1$

$$P_{a\to a+b} = \int_a^{a+b} \psi^*(x)\cdot\psi(x)\cdot dx = \int_a^{a+b} \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)\cdot dx$$

$$= \frac{1}{L}\int_a^{a+b} 1 - \cos\left(\frac{2n\pi x}{L}\right)\cdot dx$$

$$= \frac{1}{L}\left[b - \frac{\cancel{2\cos\left(\frac{2n\pi}{L}\left(a+\frac{b}{2}\right)\right)\sin\left(\frac{2n\pi}{L}\cdot\frac{b}{2}\right)}}{2n\pi/L}\right] \to 0$$

$\frac{\sin(x)}{2x}$ $\quad \frac{b}{2}$ $\quad \left(\lim_{x\to\infty} \frac{b}{2}\frac{\sin x}{x} = 0\right)$

at higher energies (i.e. for $n \to \infty$)

$P_{a\to a+b} \to \boxed{b/L} \to$ as expected classically

5. * Consider a one-dimensional particle which is confined within the region $0 \le x \le a$ and whose wave function is $\Psi(x,t) = \sin(\pi x/a)\exp(-i\omega t)$. $\longrightarrow$ stationary state with energy $(\hbar\omega)$
(a) Find the potential $V(x)$.
(b) Calculate the probability of finding the particle in the interval $a/4 \le x \le 3a/4$.

$i\hbar\cdot\frac{\partial}{\partial t}\psi(x,t) = E\psi(x,t)$

$E = \hbar\omega$

Sol^n (a) $\psi(0,t) = 0$ $\qquad V(x) = \infty \quad x > a$

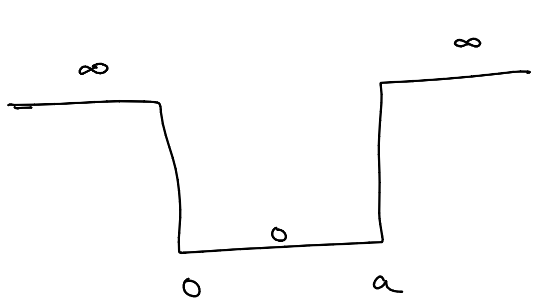
$\psi(a,t) = 0$ $\qquad V(x) = \infty \quad$ or $x < 0$

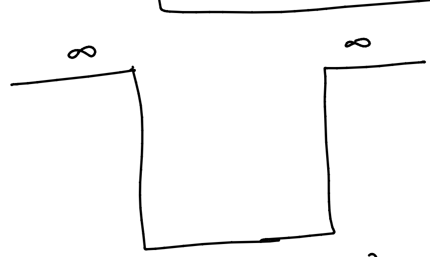
for $0 < x < a$ TDSE $\quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$

$$\neq -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \quad (\hbar\omega)$$

$$+\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2\psi(x,t) + V(x)\cdot\psi(x,t) = \hbar\omega\,\psi(x,t)$$

$$\boxed{V(x) = \hbar\omega - \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2}$$





$\hbar\omega - \frac{\hbar}{2m}(\pi/a)^2$

$V(x) = 0 \Rightarrow \hbar\omega = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2$

$\hbar\omega = \frac{h^2}{8ma^2} = E_n$

$$(b)\quad P_{a/4\to 3a/4} = \frac{\int_{a/4}^{3a/4} \psi^*(x,t)\cdot\psi(x,t)\cdot dx}{\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\cdot dx} = \frac{\int_{a/4}^{3a/4} \sin^2\left(\frac{\pi x}{a}\right)\cdot dx}{(a/2)}$$

$\sqrt{\frac{2}{a}}$

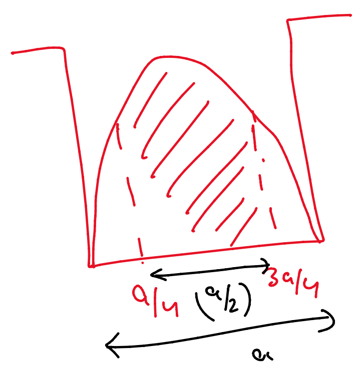
$$\int_{-\infty}^{\infty} \psi^*(x,t)\,\psi(x,t)\cdot dx$$

$(a/2)$

$$= \frac{1}{a}\left[\int_{a/4}^{3a/4} \left(1-\cos\left(\frac{2\pi x}{a}\right)\right)\right]$$

$$= \frac{1}{a}\left[\frac{a}{2} - \frac{(-2)}{(2\pi/a)}\right]$$

$$\leftarrow \quad = \frac{1}{2} + \frac{1}{\pi} > \frac{1}{2}$$



$\frac{a/2}{a} = \frac{1}{2} \rightarrow$ classically

$n \rightarrow 2m \quad (m \neq 0)$

given $\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\cdot dx = 0$

$\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\cdot dx = 0$

8. * Consider a particle of mass m in an infinite potential well extending from $x = 0$ to $x = L$. Wave function of the particle is given by

$$\psi(x) = A\left[\underbrace{\sin\left(\frac{\pi x}{L}\right)}_{\text{1st state POB}} + \overbrace{\sin\left(\frac{2\pi x}{L}\right)}^{\text{2nd state}}\right]$$

where A is the normalization constant

(a) Calculate A

(b) Calculate the expectation values of x and x^2 and hence the uncertainty Δx.

(c) Calculate the expectation values of p and p^2 and hence the uncertaintyΔp.

(d) What is the probability of finding the particle in the first excited state, if an energy measurement is made?

Sol^n $\int_0^L \psi^*(x)\,\psi(x)\cdot dx = 1$

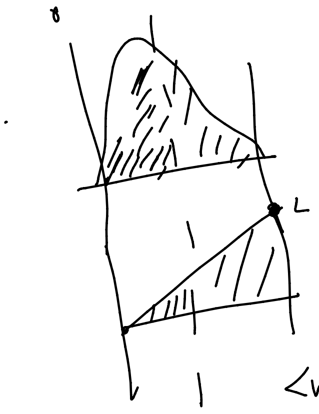
$$\int_0^L A^2\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + \cancel{2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)}\right]\cdot dx = 1$$

$\rightarrow 0$ (why?)

$$A^2\cdot\left[\frac{L}{2} + \frac{L}{2}\right] = 1 \Rightarrow \boxed{A = \frac{1}{\sqrt{L}}}$$

(b) $\langle x\rangle = \int_0^L x\cdot\psi(x)\cdot\psi^*(x)\cdot dx = \int_0^L \frac{1}{L}\cdot x\cdot\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\cdot dx$

$$= \frac{1}{2L}\int_0^L x\left(1-\cos\frac{2\pi x}{L}\right) + x\left(1-\cos\frac{4\pi x}{L}\right) + 2x\left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right]$$



$$= \frac{1}{2L}\left(\frac{L^2}{2} + \frac{L^2}{2}\right) = \frac{L}{2}$$

$$\langle x^2\rangle = \frac{1}{2L}\int_0^L x^2\left(1-\cos\frac{2\pi x}{L}\right) + x^2\left(1-\cos\frac{4\pi x}{L}\right) + 2x^2\left[\cos\frac{\pi x}{L} - \cos\left(\frac{3\pi x}{L}\right)\right]$$

$$= \frac{1}{2L}\left[\frac{L^3}{3} + \frac{L^3}{3}\right] = \frac{L^2}{3}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{L}{\sqrt{\ }}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{L}{\sqrt{12}}$$

(c) $$\langle p\rangle = \int_0^L \varphi^*(x)\left(-i\hbar \cdot \frac{\partial \varphi(x)}{\partial x}\right).dx = \frac{1}{L}\int_0^L \left(\sin\frac{2\pi x}{L} + \sin\frac{\pi x}{L}\right)\left[\frac{-2\pi i\hbar}{L}\cos\frac{2\pi x}{L} - \frac{\pi i\hbar}{L}\cos\frac{\pi x}{L}\right].dx$$

$$= \frac{(-i\hbar)}{L}\left(\frac{\pi}{L}\right)\int_0^L \cancel{2\sin\frac{2\pi x}{L}\cos\frac{2\pi x}{L}}^{\,0} + 2\sin\frac{\pi x}{L}\cos\frac{2\pi x}{L} + \cancel{\sin\frac{\pi x}{L}\cos\frac{\pi x}{L}}^{\,0} + \sin\frac{2\pi x}{L}\cos\frac{\pi x}{L}.dx$$

$$= \frac{(-i\hbar)\pi}{L^2}\int_0^L \sin\left(\frac{3\pi x}{L}\right) - \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\left[\sin\frac{3\pi x}{L} + \sin\frac{\pi x}{L}\right]$$

$$= \frac{(-i\hbar)\pi}{L^2}\left[\frac{2}{3\pi/L} - \frac{2}{\pi/L} + \frac{1}{2}\left[\frac{2}{3\pi/L} + \frac{2}{\pi/L}\right]\right]$$

$$= \frac{(-i\hbar)\pi}{L^2}\left[\frac{L}{\pi} - \frac{L}{\pi}\right] = 0$$

$$\langle p^2\rangle = \int_0^L \varphi^*(x)\left[-\hbar^2\frac{\partial^2\varphi(x)}{\partial x^2}\right] = \frac{+\hbar^2}{L}\left(\frac{\pi}{L}\right)^2\int_0^L\left(\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L}\right)\left(\sin\frac{\pi x}{L} + 4\sin\frac{2\pi x}{L}\right).dx$$

$$= \frac{\hbar^2}{L}\left(\frac{\pi}{L}\right)^2\int_0^L\left(\sin^2\frac{\pi x}{L} + 4\sin^2\frac{2\pi x}{L} + 5\sin\frac{\pi x}{L}\cdot\sin\frac{2\pi x}{L}\right).dx$$

$$= \frac{\hbar^2}{L}\left(\frac{\pi}{L}\right)^2\int_0^L\frac{\left(1-\cos\frac{2\pi x}{L}\right)}{2} + 2\left(1-\cos\frac{4\pi x}{L}\right) + \frac{5}{2}\left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right).dx$$

$$= \frac{\hbar^2}{L}\left(\frac{\pi}{L}\right)^2\left[\frac{L}{2} + 2L\right] = \frac{5}{2}\left(\frac{\hbar\pi}{L}\right)^2$$

10. * An electron is bound in an infinite potential well extending from $x = 0$ to $x = L$. At time $t = 0$, its normalized wave function is given by

$$\psi(x,0) = \frac{2}{\sqrt{L}} \sin\left(\frac{3\pi x}{2L}\right) \cos\left(\frac{\pi x}{2L}\right)$$

(a) Calculate $\psi(x,t)$ at a later time t.

(b) Calculate the probability of finding the electron between $x = L/4$ and $x = L/2$ at time t.

$E_1 = \frac{h^2}{8mL^2}$ $\quad E_2 = \frac{h^2}{2mL}$

Sol^n $\quad \varphi(x,0) = \frac{1}{\sqrt{L}}\left[\underbrace{\sin\left(\frac{2\pi x}{L}\right)}_{2^{nd}\text{ state}} + \underbrace{\sin\left(\frac{\pi x}{L}\right)}_{1^{st}\text{ state}}\right]$

$$= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) + \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right] = \frac{1}{\sqrt{2}}\left(\Phi_1(x) + \Phi_2(x)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\Phi_1(x,t) + \Phi_2(x,t)\right)$$

$$\varphi(x,t) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)e^{-i\frac{h^2/8mL^2\,t}{\hbar}} + \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)e^{-i\cdot\frac{h^2/2mL^2\,t}{\hbar}}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\Phi_2(x)e^{-iE_2t/\hbar} + \Phi_1(x)e^{-iE_1t/\hbar}\right]$$

$$P_{L/4\to L/2} = \int_{L/4}^{L/2} \varphi(x,t)\cdot\varphi^*(x,t)\cdot dx$$

$$= \int_{L/4}^{L/2}\frac{2}{L}\frac{1}{2}\left[\sin^2\left(\frac{2\pi x}{L}\right) + \sin^2\left(\frac{\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\cdot dx$$

$$= \frac{1}{L}\int_{L/4}^{L/2}\frac{\left(1-\cos\frac{4\pi x}{L}\right)}{2} + \frac{\left(1-\cos\left(\frac{2\pi x}{L}\right)\right)}{2} + \cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\cdot dx$$

$$= \frac{1}{L}\left[\frac{L/4}{2} + \frac{L/4}{2} - \frac{(-1)}{4\pi/L} + \frac{(1-1/\sqrt{2})}{(\pi/L)} - \frac{(-1-1/\sqrt{2})}{(3\pi/L)}\right]$$

$$= \frac{1}{L}\left[\frac{L}{4} + \frac{L}{4\pi} + \frac{L}{\pi}\left(1+\frac{1}{3}\right) + \frac{L}{\pi}\left(\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{L}{4} + \frac{L}{4\pi} + \frac{4L}{3\pi} - \frac{2L}{3\pi\sqrt{2}}\right]$$

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9. Suppose we have 10,000 rigid boxes of same length L from $x = 0$ to $x = L$. Each box contains one particle of mass m. All these particles are in the ground state.

(a) If a measurement of position of the particle is made in all the boxes at the same time, in how many of them, the particle is expected to be found between $x = 0$ and $L/4$?

(b) In a particular box, the particle was found to be between $x = 0$ and $L/4$. Another measurement of the position of the particle is carried out in this box immediately after the first measurement. What is the probability that the particle is again found between $x = 0$ and $L/4$?

Sol^n (a) $\quad P_{x=0\to x=L/4} = \int_0^{L/4}\frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\cdot dx = \frac{(b)}{1000}$

(b) $\varphi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) = a_1 \underbrace{u_1(x)}_{(0-L/4)} + a_2 \underbrace{u_2(x)}_{(4x, L)}$

6. An electron is moving freely inside a one-dimensional infinite potential box with walls at $x = 0$ and $x = a$. If the electron is initially in the ground state ($n = 1$) of the box and if we suddenly quadruple the size of the box (i.e., the right-hand side wall is moved instantaneously from $x = a$ to $x = 4a$), calculate the probability of finding the electron in:

(a) the ground state of the new box and

(b) the first excited state of the new box.

Sol^n $\varphi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{a}\right) = \sum_{i=1}^{\infty} a_i \underbrace{\phi_i(x)}$

wave functions corresponding to the new box

$$\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{a}\right) = \sum_{i=1}^{a} a_i \sin\left(\frac{n\pi x}{4a}\right)$$

$$\int_0^{4a} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \cdot \sin\left(\frac{\pi x}{4a}\right) = \sum_{i=1}^{a} \int_0^{4a} a_i \sin\left(\frac{i\pi x}{4a}\right)\left(\sin\left(\frac{\pi x}{4a}\right)\right)$$

$$\sqrt{\frac{2}{a}}\left[\frac{1}{2} \int_0^{4a} \left(\cos\frac{3\pi x}{4a} - \cos\frac{5\pi x}{4a}\right)\right] = a_1\left(\frac{4a}{2}\right)$$

$$\sqrt{\frac{2}{a}} \cdot \frac{1}{2}\left[\frac{x}{3\pi/4a} - \frac{x}{5\pi/4a}\right] = a_1\left(\frac{4a}{2}\right)$$

$$\sqrt{\frac{2}{a}} \frac{(2\pi/4a)}{\frac{15\pi^2}{16a^2}} = a_1\left(\frac{4a}{2}\right)$$

$$a_1 = \sqrt{\frac{2}{a}} \frac{4\pi/16a^2}{15\pi^2/16a^2} = \frac{4}{15\pi}\sqrt{\frac{2}{a}}$$

$$\frac{|a_1|^2}{\sum |a_i|^2}$$

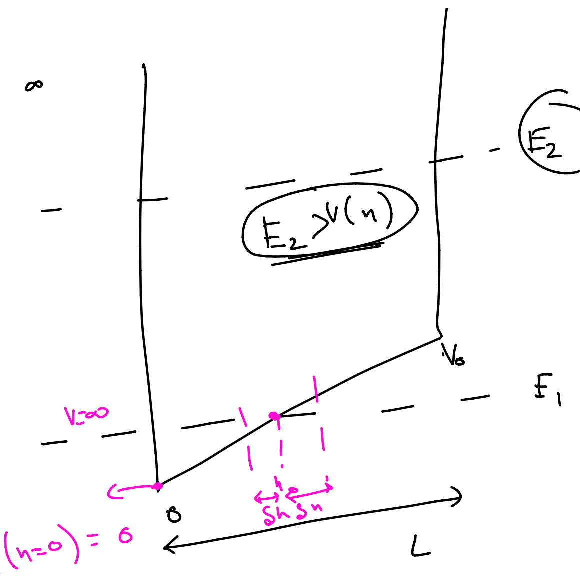
(11) (a) $1\,\mu J = \frac{n^2 h^2}{8 m L^2}$ $\qquad L = 1\,\mu m$ $\qquad m = 1\,\mu g$

$$1 \times 10^{-6} = \frac{n^2 (6.63^2 \times 10^{-68})}{8 \times 10^{-6} \times 10^{-12}} \qquad n^2 = c\,10^{44} \qquad n = c \cdot 10^{22}$$

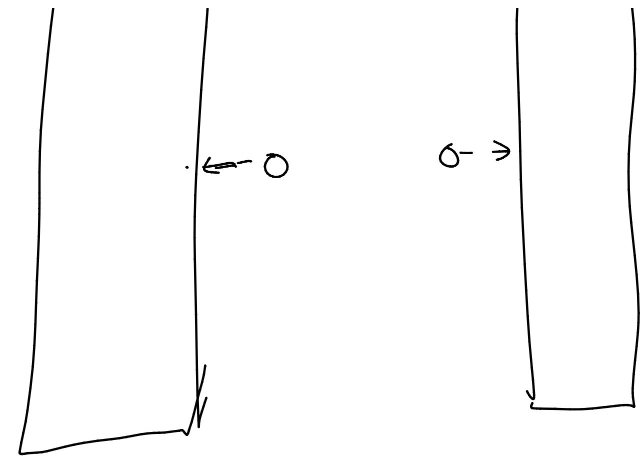
(b)

(12)

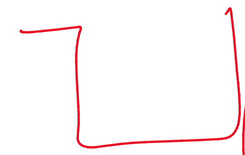
(a)



$\psi(x=0) = 0$



$V(x)$

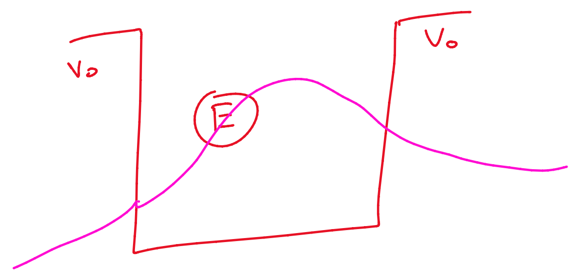


$\sin\left(\frac{\pi x}{L}\right)$

$\sin\left(\frac{2\pi x}{L}\right)$

$0 < x < x_0$ $\quad V(x) < E_1$

$L > x > x_0$ $\quad V(x) > E_1$



$E > V(x) \rightarrow$

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\,\psi(x) = E\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = [E - V(x)]\,\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\,\psi(x)$$

$0 < x < x_0 - \delta \rightarrow$ sinusoidal

$L \quad x > x_0 + \delta \rightarrow$ decays

$x_0 - \delta < x < x_0 + \delta$

$V(x)$

$E > V(x)$ eigenvalue (−)

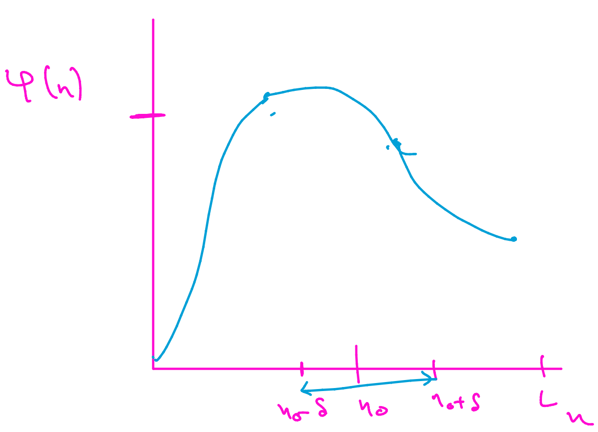
$E < V(x)$ eigenvalue (+)

$k = \sqrt{\frac{2m}{\hbar^2}(E - V(x))}$ ↑

$(A \sin kx + B \cos kx)$

$(Ce^{\alpha x} + De^{-\alpha x})$

$\alpha = \sqrt{\frac{2m}{\hbar^2}(V(x) - E)}$

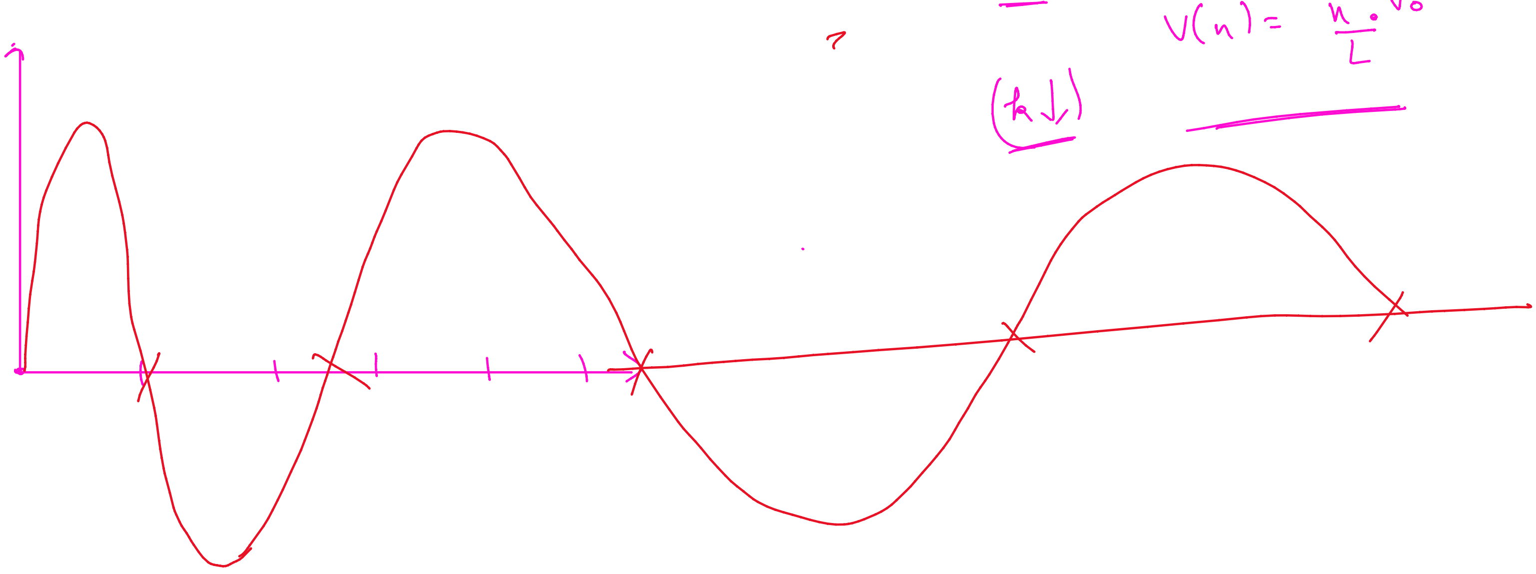


(b) $E_2 > V(x) \rightarrow$ sinusoidal

As $x \uparrow$ $\quad V(x) \uparrow$

$V(x) = \frac{x}{L} \cdot V_0$

$(k \downarrow)$



$V \rightarrow \infty$ $\quad \psi(0^-) = 0$

$\psi(0^+) = 0$

Q-2

$$N \sin^2\left(\frac{\pi x}{L}\right) = N\left(1 - \cos\frac{2\pi x}{L}\right) = \sum a_i \sin\left(\frac{i\pi x}{L}\right)$$

$$a_2 = N \int_0^L \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right) \cdot \sin\left(\frac{2\pi x}{L}\right)$$