Tutorial 8 Solution

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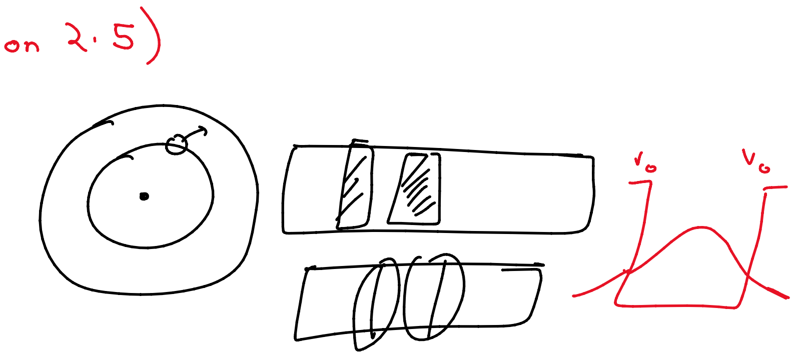
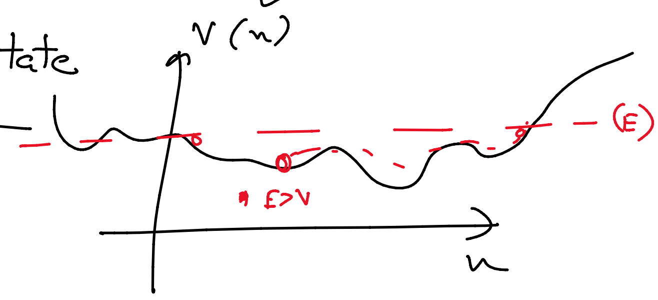
Bound States & Scattered States (Griffiths section 2.5)

$\downarrow$ Definite (discrete) energy states

$\downarrow$ Continuous energy states



Conditions for a state with energy E to be a bound state



$V(x=\infty)$ $V(x=-\infty)$

(i) $E < V(\infty)$ and $E < V(-\infty)$

$E < V_0$

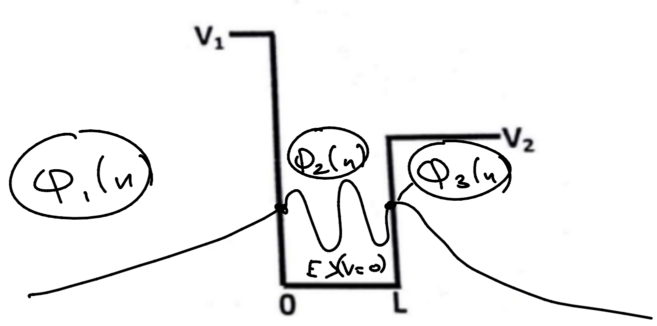
$$-\frac{\hbar^2}{2m}\psi''(x) + V_0\,\psi(x) = E\,\psi(x)$$

$$\psi(x) = (Ae^{-\alpha x} + Be^{\alpha x})$$

(ii) $E > (V_{min})$ $\rightarrow$ For normalizability

Prove that if $E \le V_{min}$ then $\psi(x)$ won't be a normalizable wave function

1. * Consider the asymmetric finite potential well of width L, with a barrier V_1 on one side and a barrier V_2 on the other side. Obtain the energy quantization condition for the bound states in such a well. From this condition derive the energy quantization conditions for a semi-infinite potential well (when $V_1 \rightarrow \infty$ and V_2 is finite).



Sol^n (i) $E < V(\infty) \Rightarrow E < V_2$ (ii) $E < V(-\infty) \Rightarrow E < V_1$ (iii) $E > V_{min} = 0$

$$-\frac{\hbar^2}{2m}\phi_1''(x) + V_1\phi_1(x) = E\,\phi_1(x)$$

$$\frac{\hbar^2}{2m}\phi_1''(x) = \underbrace{(V_1 - E)}_{>0}\,\phi(x) \qquad \alpha \in R$$

$$\phi_1(x) = Ae^{\alpha_1 x} + (Be^{-\alpha_1 x}) \qquad \alpha_1 = \sqrt{\frac{2m(V_1-E)}{\hbar^2}}$$

$x \to -\infty$ $\phi_1(x) \to 0$

$\Rightarrow B = 0$

$$\phi_1(x) = Ae^{\alpha_1 x}$$

$$\phi_2(x) = Ce^{ikx} + De^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\phi_3(x) = Fe^{-\alpha_2 x} \qquad \alpha_2 = \sqrt{\frac{2m(V_2-E)}{\hbar^2}}$$

$$\varphi_3(x) = Fe^{-\alpha_2 x} \qquad \alpha_2 = \sqrt{\frac{2m(V_2-E)}{\hbar^2}}$$

$$\varphi_1(0) = \varphi_2(0) \qquad \varphi_2(L) = \varphi_3(L)$$

$$A = C + D \quad -① \qquad Ce^{ikL} + De^{-ikL} = Fe^{-\alpha_2 L} \quad -②$$

$$\varphi_1'(0) = \varphi_2'(0) \qquad \varphi_2'(L) = \varphi_3'(L)$$

$$\alpha_1 A = ik(C-D) \quad -③ \qquad ik\left(Ce^{ikL} - De^{-ikL}\right) = -\alpha_2 Fe^{-\alpha_2 L}$$

$$Ce^{ikL} - De^{-ikL} = \frac{-\alpha_2 Fe^{-\alpha_2 L}}{ik} \quad -④$$

$$A\left(1 + \frac{\alpha_1}{ik}\right) = 2C \quad -⑤$$

$$A\left(1 - \frac{\alpha_1}{ik}\right) = 2D \quad -⑥$$

$$②+④ \Rightarrow 2Ce^{ikL} = Fe^{-\alpha_2 L}\left(1 - \frac{\alpha_2}{ik}\right) \quad -⑦$$

$$②-④ \Rightarrow 2De^{-ikL} = Fe^{-\alpha_2 L}\left(1 + \frac{\alpha_2}{ik}\right) \quad -⑧$$

$$5/6 \Rightarrow \frac{1 + \alpha_1/ik}{1 - \alpha_1/ik} = \frac{C}{D} \qquad 7/8 \Rightarrow \frac{C}{D} = \frac{(1-\alpha_2/ik)}{(1+\alpha_2/ik)} e^{-2ikL}$$

$$\Rightarrow \frac{1 + \alpha_1/ik}{1 - \alpha_1/ik} = \frac{1 - \alpha_2/ik}{1 + \alpha_2/ik} e^{-2ikL}$$

$$\Rightarrow \frac{\left(1 - \frac{\alpha_1\alpha_2}{k^2}\right) - i\left(\frac{\alpha_1}{k} + \frac{\alpha_2}{k}\right)}{\left(1 - \frac{\alpha_1\alpha_2}{k^2}\right) + i\left(\frac{\alpha_1}{k} + \frac{\alpha_2}{k}\right)} = \frac{e^{-ikL}}{e^{+ikL}}$$

$\Rightarrow$ C & D

$$\frac{\left(1 - \frac{\alpha_1\alpha_2}{k^2}\right)}{\cancel{i}\left(\frac{\alpha_1}{k} + \frac{\alpha_2}{k}\right)} = \frac{\cos kL}{\cancel{i}\sin kL}$$

$$\Rightarrow \tan kL = \frac{\frac{\alpha_1}{k} + \frac{\alpha_2}{k}}{1 - \frac{\alpha_1\alpha_2}{k^2}} \longrightarrow \underline{\text{Solve graphically}}$$

... finite well | Finite square box

→ Semi infinite well

$V_1 \to \infty$

$\Rightarrow \alpha_1 = \infty$

$\tan kL = -\frac{k}{\alpha_2}$

Finite square box

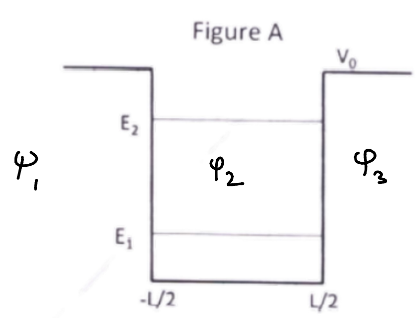
$V_1 = V_2 = V_0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_0$

$$\boxed{\tan kL = \frac{2\alpha_0 k}{k^2 - \alpha_0^2}}$$

3

3. An electron is trapped in a 1-dimensional symmetric potential well of height V_0 and width L (Figure A). The energy of electron is E, its wave number inside the well is k, and the magnitude of the wave number outside the well is α.

Figure A



(0 L)

(-L/2 L/2)

(a) Derive the energy quantization conditions, in terms of k, α and L, for the symmetric and anti-symmetric bound states.

(b) When the width of the well is L = 0.2 nm, it is found that the ground state energy $E_1 = 4.45$eV, and the first excited state energy $E_2 = 15.88$eV. Calculate V_0. ($E_2 = 4E_1$)

(c) Calculate the penetration depth for the ground state.

(d) If the width of the potential well is doubled to 2 L keeping V_0 the same, estimate the change in the ground state energy.

(e) Consider the potential which is generated from Figure A by setting V = ∞ at L = 0. What is the energy of the ground state in this case?

(f) How many bound states are possible in this case?

Sol^n

$$-\frac{\hbar^2}{2m}\varphi_1''(x) + V_0\varphi_1(x) = E\varphi_1(x) \qquad E<V_0 \text{ (Bound states)}$$

$$-\frac{\hbar^2}{2m}\varphi_2''(x) = E\varphi_2(x) \qquad E>0$$

$$-\frac{\hbar^2}{2m}\varphi_3''(x) + V_0\varphi_3(x) = E\varphi_3(x) \qquad E<V_0$$

$$\alpha = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$\varphi_1(x) = Ae^{\alpha x}$

$\varphi_2(x) = Ce^{ikx} + De^{-ikx}$

$\varphi_3(x) = Be^{-\alpha x}$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$\varphi_2(L/2) = \varphi_3(L/2)$

$\varphi_1(-L/2) = \varphi_2(-L/2)$

$$\varphi_1(-L/2) = \varphi_2(-L/2)$$

$$\varphi_2(L/2) = \varphi_3(L/2)$$

$$Ae^{-\alpha L/2} = Ce^{-ikL/2} + De^{+ikL/2} \quad (1)$$

$$Ce^{ikL/2} + De^{-ikL/2} = Be^{+\alpha L/2} \quad - (3)$$

$$\varphi_2'(L/2) = \varphi_3'(L/2)$$

$$\varphi_1'(-L/2) = \varphi_2'(L/2)$$

$$ik\left(Ce^{ikL/2} - De^{ikL/2}\right) = -\alpha Be^{+\alpha L/2} \quad - (4)$$

$$+\alpha Ae^{-\alpha L/2} = ik\left(Ce^{-ikL/2} - De^{+ikL/2}\right) \quad (2)$$

$$C = \frac{Be^{-\alpha L/2}}{2}\left(1 - \frac{\alpha}{ik}\right)e^{-ikL/2}$$

$$C = \frac{Ae^{-\alpha L/2}}{2}\left(1 + \frac{\alpha}{ik}\right)\cdot e^{ikL/2}$$

$$D = \frac{Be^{-\alpha L/2}}{2}\left(1 + \frac{\alpha}{ik}\right)e^{ikL/2}$$

$$D = \frac{Ae^{-\alpha L/2}}{2}\left(1 - \frac{\alpha}{ik}\right)e^{-ikL/2}$$

$$\frac{C}{D} = \frac{e^{-ikL/2}}{e^{ikL/2}}\,\frac{\left(1 - \frac{\alpha}{ik}\right)}{\left(1 + \frac{\alpha}{ik}\right)}$$

$$\frac{C}{D} = \frac{e^{ikL/2}}{e^{-ikL/2}}\cdot\left(\frac{1 + \alpha/ik}{1 - \alpha/ik}\right)$$

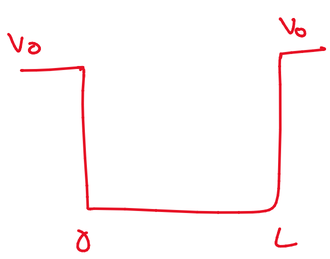
$$e^{ikL}\left(1 + \frac{\alpha}{ik}\right)^2 = e^{-ikL}\left(1 - \frac{\alpha}{ik}\right)^2$$

$$\frac{e^{ikL}}{e^{-ikL}} = \frac{\left(1 - \frac{\alpha}{ik}\right)^2}{\left(1 + \frac{\alpha}{ik}\right)^2} = \frac{\left(1 - \alpha^2/k^2\right) + 2i\frac{\alpha}{k}}{\left(1 - \frac{\alpha^2}{k^2}\right) - 2i\frac{\alpha}{k}}$$

$$\frac{\cos kL}{i\sin kL} = \frac{\left(1 - \alpha^2/k^2\right)}{+2i\alpha/k}$$

$$\boxed{\tan kL = \frac{2\alpha k}{k^2 - \alpha^2}}$$

→ Same as the one obtained for



$$\tan kL = \frac{2\,\alpha/k}{1 - \frac{\alpha^2}{k^2}} = \frac{2\sqrt{\frac{V_0 - E}{E}}}{1 - \frac{V_0 - E}{E}} = \frac{2\sqrt{E(V_0 - E)}}{2E - V_0}$$

$$\underline{\tan\left(\sqrt{\frac{2mE}{\hbar^2}}\cdot L\right)} = \frac{2\sqrt{E(V_0 - E)}}{2E - V_0}$$

(b) $E_n = \dfrac{n^2 h^2}{8m(L + 2\delta_n)^2} \longleftrightarrow \quad \delta_n = \dfrac{1}{\sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}}$

(4) 4. *Consider a particle of mass m in a potential given by

$$\begin{aligned} V(x) &= 0 \text{ for } |x| < L/2, \\ &= V_0 \text{ for } L/2 < |x| < L \\ &= \infty \text{ for } |x| \geq L \end{aligned}$$

(4) 4. *Consider a particle of mass m in a potential given by

$$V(x) = 0 \text{ for } |x| < L/2,$$
$$= V_0 \text{ for } L/2 < |x| < L$$
$$= \infty \text{ for } |x| \geq L$$

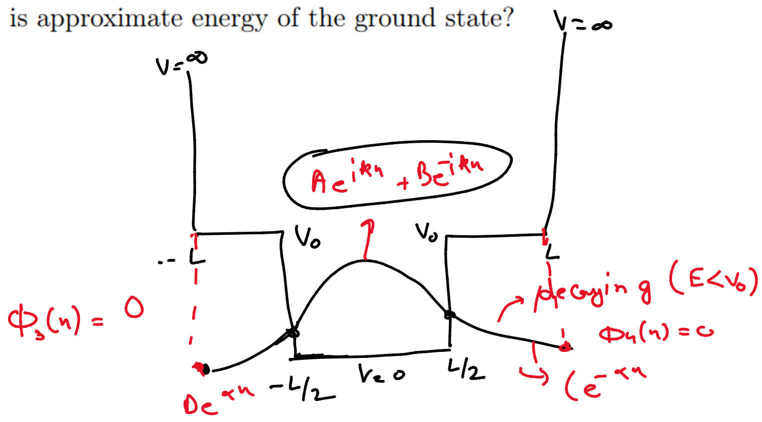
(a) Sketch the potential and the qualitative nature of the ground-state wave-function (without solving the Schrodinger equation). Mention the functional form of the wave function in each region.

(b) An electron is trapped in a symmetric finite potential of depth $V_0 = 1000$eV and width $L = 1$. What is approximate energy of the ground state? 1Å

(i) $(E < \infty,\ E < \infty,\ E > 0)$ $(E > V_0)$

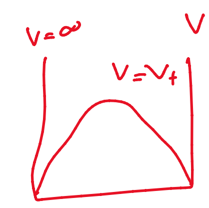
Sol^n

(a)

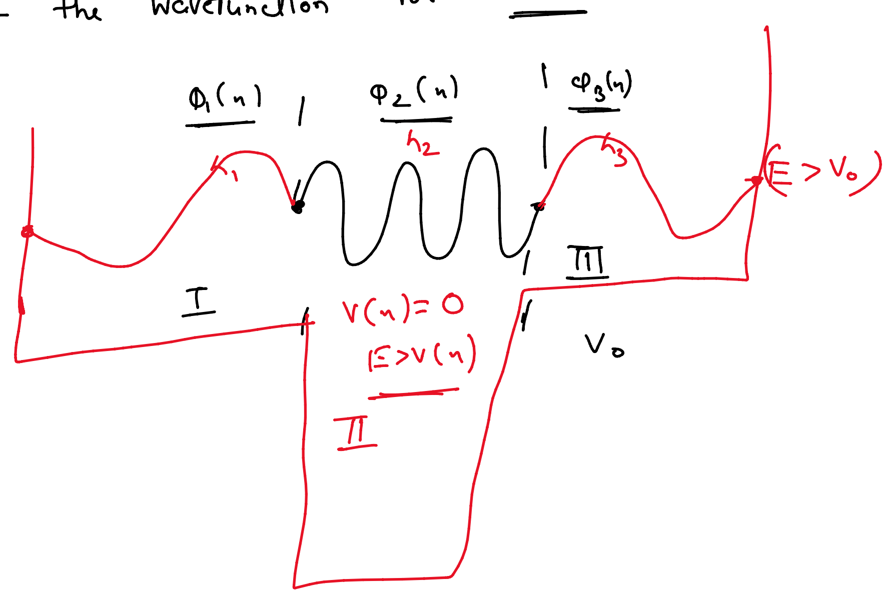


Ground state → $(E < V_0)$ (lowest energy bound state)

No Nodes



→ Plot the wavefunction for $E > V_0$



$\lambda_1 = \lambda_3 > \lambda_2$

$k_1 \quad k_3 \quad k_2$

$k_2 = \sqrt{\frac{2mE}{\hbar^2}} > k_1 = k_3 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$\lambda_2 < (\lambda_1 = \lambda_3)$

(b) $E_n = \frac{n^2h^2}{8m(L+2\delta_n)^2}$ ↳ penetration depth $= \frac{1}{\alpha_n}$

$\delta_n = \frac{1}{\alpha_n} = \sqrt{\frac{\hbar^2}{2m(V_0-E)}} \sim \sqrt{\frac{(0.197 \times 10^{-6})^2}{2 \times 0.511 \times 10^6 \times 1000}} \sim 10^{-12}$

$\alpha_n = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

10^{-10}

$E_n = \frac{n^2h^2}{8mL^2}\left(1 - \frac{4\delta_n}{L}\right)$

$\delta_n = \sqrt{\frac{\hbar^2}{2m(V_0-E_n)}} = \sqrt{\frac{\hbar^2}{2mV_0}}\left(1 + \frac{E_n}{2V_0}\right)$

$\frac{8mL^2}{h^2}E_n = 1 - \frac{4}{L}\sqrt{\frac{\hbar^2}{2mV_0}}\left(1 + \frac{E_n}{2V_0}\right)$

$E_n\left(\frac{8mL^2}{h^2} - \frac{2}{L}\sqrt{\frac{\hbar^2}{2mV_0}} \cdot \frac{1}{V_0}\right) = \left(1 - \frac{4}{L}\sqrt{\frac{\hbar^2}{2mV_0}}\right)$

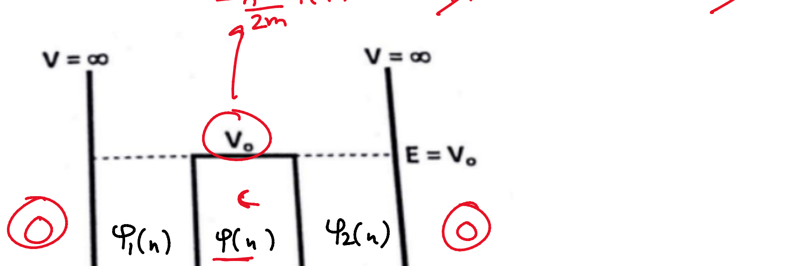
10^{-2} $\sim 10^{-10}$ $\sim 10^{-12}$ 10^3 10^{-5} 10^{-2}

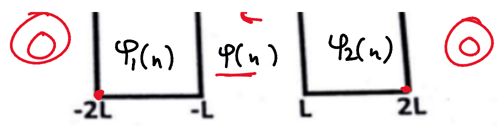
$E_n \simeq \frac{n^2h^2}{8mL^2}$

(6)

6. *A particle of mass m is bound in a double well potential shown in the figure. Itsenergy eigenstate $\psi(x)$ has energy eigenvalue $E = V$ o (where V$_o$ is the energy of the plateau in the middle of the potential well). It is known that $\psi(x) = C$ (C is a constant) in the plateau region.

$-\frac{\hbar^2}{2m}\psi(x) + \cancel{V_0\psi(x)} = E\psi(x) = \cancel{V_0\psi(x)}$





(a) Obtain $\psi(x)$ for the regions $-2\,\mathrm{L} < \mathrm{x} < -\mathrm{L}$ and $L < x < 2L$ and the relation between the wavenumber ' k ' and L.
(b) Determine ' C ' in terms of L.
(c) Assume that the bound particle is an electron and $L = 1\mathring{A}$. Calculate the 2 lowest values of V_o (in eV) for which such a solution exists.
(d) For the smallest allowed k, calculate the expectation values for x, x^2, p and p^2 and show that Heisenberg's Uncertainty Relation is obeyed.

Sol^n (a) $$-\frac{\hbar^2}{2m}\varphi_1(x) = E\,\varphi_1(x)$$

$$\varphi_1(x) = Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{\frac{2mV_o}{\hbar^2}}$$

Similarly $$\varphi_2(x) = E e^{ikx} + F e^{-ikx}$$

$$\varphi_1(-L) = \varphi(-L) \qquad \varphi_2(L) = \varphi(L)$$

$$Ae^{-ikL} + Be^{ikL} = C \qquad Ee^{ikL} + Fe^{-ikL} = C$$

$$\varphi_1'(-L) = \varphi'(-L) \qquad \varphi_2'(L) = \varphi'(L)$$

$$ik\left(Ae^{-ikL} - Be^{ikL}\right) = 0 \qquad ik\left(Ee^{ikL} - Fe^{ikL}\right) = 0$$

$$A = \frac{C\cdot e^{ikL}}{2} \qquad E = \frac{C}{2}e^{-ikL}$$

$$B = \frac{C}{2}e^{-ikL} \qquad F = \frac{C}{2}e^{ikL}$$

$$\varphi_1(x) = \frac{C}{2}e^{ik(x+L)} + \frac{C}{2}e^{-ik(x+L)} = C\cos k(x+L)$$

$$\varphi_2(x) = \frac{C}{2}e^{+ik(x-L)} + \frac{C}{2}e^{ik(L-x)} = C\cos k(x-L)$$

Also $$\varphi_1(-2L) = 0 \qquad \varphi_2(2L) = 0$$

$$\cos -kL = 0 \qquad \cos kL = 0$$

$$\boxed{kL = (2n+1)\frac{\pi}{2}} = \sqrt{\frac{2mV_o}{\hbar^2}}\cdot L = (2n+1)\frac{\pi}{2} \qquad n > 0$$

(b) Using normalizability

$$\int_{-2L}^{-L} C^2\cos^2 k(x+L)\cdot dx + \int_{-L}^{L} C^2\cdot dx + \int_{L}^{2L} C^2\cos^2 k(x-L)\cdot dx = 1$$

$$C^2\int_{-L}^{0}\cos^2 kt\cdot dt + \int_{-L}^{L} C^2\cdot dx + C^2\int_{0}^{L}\cos^2 kt\cdot dt = 1$$

$$C^2\int^{L}\cos^2 k\,t\cdot dt + 2LC^2 = 1$$

$$c^2 \int_{-L}^{L} \cos^2 kt \, dt + 2Lc^2 = 1$$

$$\frac{c^2}{2} \int_{-L}^{L} (1 + \cos 2kt) \, dt + 2Lc^2 = 1$$

$$\frac{c^2}{2} \left(2L + \frac{0}{2k} \right) + 2Lc^2 = 1$$

$$\boxed{c = \frac{1}{\sqrt{3L}}}$$

(c) $kL = \frac{\pi}{2}, \frac{3\pi}{2}$

(d) DIY