Tutorial 9 Solutions

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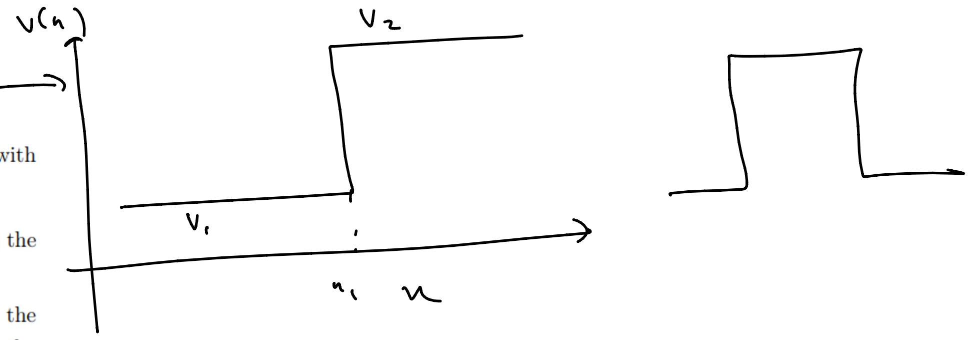


1. * A potential barrier is defined by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Particles with energy E ($< V_0$) approaches the barrier from left.

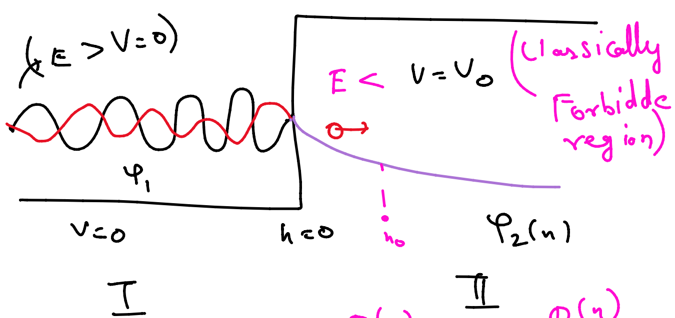
   (a) Find the value of $x = x_0$ ($x_0 > 0$), for which the probability density is $1/e$ times the probability density at $x = 0$.

   (b) Take the maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_0. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_0.

→ No

Sol^n

(a)



$\varphi_2(0) = C$

$\varphi_2(x) = Ce^{-\alpha x}$

$|\varphi_2(x)|^2 = C^2 e^{-2\alpha x}$

$\alpha = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}$

$|\varphi_2(x_0)|^2 = \dfrac{1}{e}|\varphi_2(0)|^2$

$C^2 e^{-2\alpha x_0} = \dfrac{C^2}{e}$

$\alpha x_0 = 1/2$

$$\boxed{x_0 = \frac{1}{2\alpha} = \frac{1}{2}\sqrt{\frac{\hbar^2}{2m(V_0 - E)}}}$$

$V_0 - E >>$

x_0

$\varphi_1(x) = Ae^{ikx} + Be^{-ikx}$; $k = \sqrt{\dfrac{2mE}{\hbar^2}}$

$\Phi_1(x)$ $\Phi_2(x)$

$\varphi_1(0) = A + B$

$-i\hbar \dfrac{\partial \Phi_1(x)}{\partial x} = -i\hbar (ik)\Phi_1(x)$

$= \hbar k$

$\varphi_1(0) = \varphi_2(0)$

$A + B = C$

$\varphi_1^*(0) = \varphi_2^*(0)$

$A^* + B^* = C$

$\varphi_1(0) = \varphi_2(0)$

$|\varphi_1(0)|^2 = |\varphi_2(0)|^2$

$\varphi_1(x)$

$\Leftarrow (Ae^{ikx} + Be^{-ikx})(A^*e^{-ikx} + B^*e^{ikx})$

$(A+B)(A^* + B^*)$

$|A|^2 + |B|^2 + AB^* e^{2ikx} + A^*B e^{-2ikx}$

$(A+B)^2$

$|A|^2 + |B|^2 + AB^* + BA^* \neq C^2$

(b) $\Delta x \cdot \Delta p > \dfrac{\hbar}{2}$

$\Delta p > \dfrac{\hbar}{2x_0} = \sqrt{2m(V_0 - E)}$

$\Delta E = \dfrac{(\Delta p)^2}{2m} > (V_0 - E)$

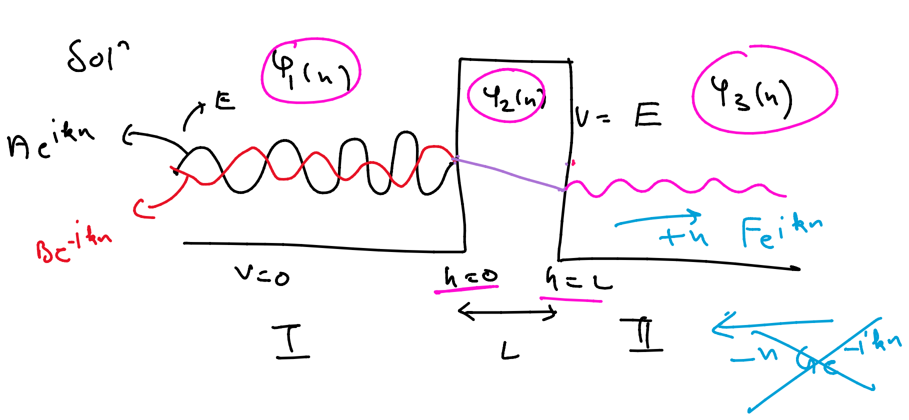
$E + \Delta E > V_0$

4. * A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in a potential free region, encounters a one-dimensional potential barrier of height $V = E$ and width L.

   (a) Obtain an expression for the transmission coefficient.

   (b) Find the value of L (in terms of λ) for which the reflection coefficient will be half.

Sol^n



(a) $-\dfrac{\hbar^2}{2m}\dfrac{d^2\varphi_1(x)}{dx^2} = E\varphi_1(x)$

$k = \sqrt{\dfrac{2mE}{\hbar^2}} = \dfrac{2\pi}{\lambda}$

$\varphi_1(x) = Ae^{ikx} + Be^{-ikx}$

moving towards +ve x

(Check using momentum operator $-i\hbar\partial$)

$-\dfrac{\hbar^2}{2m}\dfrac{d^2\varphi_2(x)}{dx^2} = 0$

$\varphi_2(x) = Cx + D$

$\varphi_2(x) = (x + D$ momentum operator

$-i\hbar \frac{\partial}{\partial x})$

$\varphi_3(x) = Fe^{ikx} + Ge^{-ikx}$

$\varphi_2(x) = Cx + D$

$\varphi_3(x) = Fe^{ikx}$

Boundary Cond^n

$\varphi_1(0) = \varphi_2(0)$ $\qquad \varphi_1'(0) = \varphi_2'(0)$ $\qquad \varphi_2(L) = \varphi_3(L)$ $\qquad \varphi_2'(L) = \varphi_3'(L)$

$\underbrace{A+B}= D$ $\qquad ik\underbrace{(A-B)} = C$ $\qquad CL + D = Fe^{ikL}$ $\qquad C = ikFe^{ikL}$

$\frac{CL+D}{C} = \frac{1}{ik}$

$A = \frac{1}{2}\left(D + \frac{C}{ik}\right)$ $\qquad B = \frac{1}{2}\left(D - \frac{C}{ik}\right) = -\frac{CL}{2}$ $\qquad L + \frac{D}{C} = \frac{1}{ik} \Rightarrow$ $\qquad D = C\left(\frac{1}{ik} - L\right)$

$A = \frac{1}{2}\left(\frac{2C}{ik} - LC\right) = \frac{C}{2}\left(\frac{2}{ik} - L\right)$ $\qquad F = \frac{C}{ik}e^{-ikL}$

$T = \frac{|F|^2}{|A|^2} = \frac{C^2/k^2}{\frac{C^2}{4}\left(\frac{2}{ik} - L\right)\left(\frac{-2}{ik} - L\right)} = \frac{4}{k^2}\frac{1}{\left(L + \frac{2}{ik}\right)\left(L - \frac{2}{ik}\right)} = \frac{4}{k^2\left(L^2 + \frac{4}{k^2}\right)}$

$$\boxed{T = \frac{4}{L^2k^2 + 4}}$$

$R = \frac{|B|^2}{|A|^2} = \frac{(-CL/2)^2}{\frac{C^2}{4}\left(L^2 + \frac{4}{k^2}\right)} = \frac{L^2}{L^2 + \frac{4}{k^2}} = \frac{L^2k^2}{L^2k^2 + 4}$ $= \frac{1}{\frac{k^2L^2}{4} + 1}$

(b) $R + T = 1$

$R = 1 - T = 1 - \frac{4}{L^2k^2 + 4} = \frac{L^2k^2}{L^2k^2+4} = \frac{1}{2}$

$L^2k^2 = 4 \Rightarrow L = \frac{2}{k} = \frac{h}{\pi}$

$\frac{B}{A} = \left(1 - \frac{2}{ikL}\right)$

$\frac{B^*}{A^*} = \left(1 + \frac{2}{ikL}\right)$ $\qquad 1 + \frac{4}{k^2L^2} = \frac{k^2L^2 + 4}{k^2L^2}$

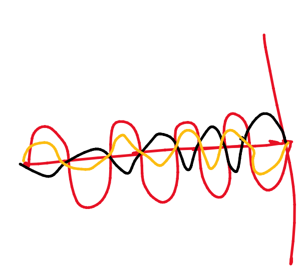
6. * A beam of particles of mass m and energy $9V_0$ (V_0 is a positive constant with the dimension of energy) is incident from left on a barrier, as shown in figure below. $V = 0$ for $x < 0$, $V = 5V_0$ for $x \le d$ and $V = nV_0$ for $x > d$. Here n is a number, positive or negative and $d = \pi\hbar/\sqrt{8mV_0}$. It is found that the transmission coefficient from $x < 0$ region to $x > d$ region is 0.75.

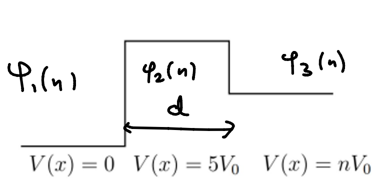
$d = \pi\hbar\sqrt{8mV_0}$ $\qquad d = \frac{\pi\hbar}{2\sqrt{2mV_0}}$ $\qquad \frac{2d}{\pi} = \frac{\hbar}{\sqrt{2mV_0}}$

(a) Find n. Are there more than one possible values for n?

(b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of n.

(c) Is there a phase change between the incident and the reflected beam at $x = 0$? If yes, determine the phase change for each possible value of n. Give your answers by explaining all the steps and clearly writing the boundary conditions used





$\boxed{n < 9}$

Sol^n (a) $\varphi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$ $\qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}} = 3\sqrt{\frac{2mV_0}{\hbar^2}} = 3k = \frac{3\pi}{h}\sqrt{8mV_0}$

Sol^n (a) $\varphi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$

$= Ae^{3ikx} + Be^{-3ikx}$

$A \cdot B < 0 \rightarrow$ phase change

$\varphi_2(x) = Ce^{2ikx} + De^{-2ikx}$

$\varphi_3(x) = Fe^{imkx}$

$k_1 = \sqrt{\frac{2mE}{\hbar^2}} = 3\sqrt{\frac{2mV_0}{\hbar^2}} = 3k = \frac{3\pi}{h}\sqrt{8mV_0}$

$\left(k = \frac{\pi}{2d}\right)$

$k = \sqrt{\frac{2mV_0}{\hbar^2}} = \frac{1}{\hbar}\sqrt{2mV_0} = \frac{\pi}{2d}$

$m = \sqrt{(9-n)}$

Boundary Cond^n

$\varphi_1(0) = \varphi_2(0)$

$A + B = C + D$

$\varphi_1'(0) = \varphi_2'(0)$

$3A - 3B = 2C - 2D$

$A - B = \frac{2}{3}(C - D)$

$\varphi_2(d) = \varphi_3(d)$

$Ce^{i\pi} + De^{-i\pi} = Fe^{mi\pi}$

$\varphi_2'(d) = \varphi_3'(d)$

$2k\left(Ce^{2i\pi/2} - De^{-2i\pi/2}\right) = mFe^{mi\pi}$

$2A = \frac{5}{3}C + \frac{1}{3}D$

$A = \frac{5}{6}C + \frac{1}{6}D$

$\frac{Ce^{i\pi} + De^{-i\pi}}{Ce^{i\pi} - De^{-i\pi}} = \frac{2}{m}$

$\frac{Ce^{i\pi}}{De^{-i\pi}} = \frac{2+m}{2-m}$

$\frac{C}{D} = \frac{2+m}{2-m}$

$A = \frac{5}{6}\left(\frac{2+m}{2-m}\right) \cdot D + \frac{1}{6}D$

$A = \frac{12+4m}{6(2-m)} \cdot D = \frac{4(3+m)}{6(2-m)} \cdot D$

$A = \frac{4(3+m)}{6(2-m)} \cdot D$

$C + D = Fe^{mi\pi}$

$\frac{4}{2-m} \cdot D = Fe^{mi\pi}$

$F = \frac{4}{2-m}(-1)^m \cdot D$

$\frac{m\cancel{k}\,|F|^2}{3\cancel{k}\,|A|^2} = T = \frac{3}{4}$

$m\left(\frac{6}{3+m}\right)^2 = \frac{9}{4}$

$\frac{4m}{(m+3)^2} = \frac{1}{4}$

$16m = m^2 + 6m + 9$

$m^2 - 10m + 9 = 0$

$\boxed{m = 1, 9}$

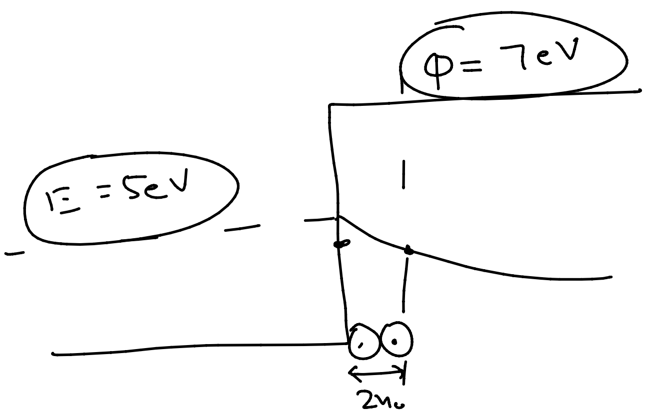
$\boxed{m = 1,9}$

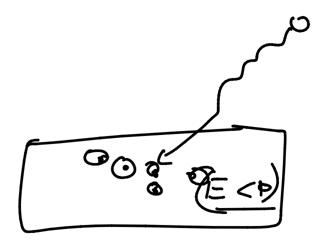
$\boxed{n = 8,72}$

(b)

(c)

7





ϕ = Potential Barrier

$I \propto |\psi_2(x)|^2$

$I \propto q_e$

$$\frac{|\psi_2(2x_0)|^2}{|\psi_2(x_0)|^2}$$

Direction of waves

$Ae^{i(kx+\omega t)}$ $Ae^{i(kx-\omega t)}$ $Ae^{i(-kx-\omega t)}$

$$-i\hbar \frac{\partial}{\partial x} Ae^{i(kx-\omega t)} = \hbar k \, Ae^{i(kx-\omega t)}$$

>0

p in +x direction