Tutorial 1 Solutions

28 March 2022 21:37

↳ Griffiths MA : 111

1.* Calculate the curl and divergence of the following vector functions. If the curl turns out to be zero, construct a scalar function $\phi$ of which the vector field is the gradient:

(a) $F_x = x + y$ ; $F_y = -x + y$ ; $F_z = -2z$

(b) $G_x = 2y$ ; $G_y = 2x + 3z$ ; $G_z = 3y$

(c) $H_x = x^2 - z^2$ ; $H_y = 2$ ; $H_z = 2xz$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$\nabla \times \vec{F} = 0 \Downarrow \boxed{\vec{F} = \nabla \phi}$$

$$\frac{\partial F_x}{\partial x} \quad \frac{\partial F_y}{\partial y} \quad \frac{\partial F_z}{\partial z}$$

(a) $\mathrm{Div}(F) = \vec{\nabla} \cdot \vec{F}$

$$= \left( \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left( F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \right)$$

$$= 1 + 1 + (-2) = 0$$

$\mathrm{Curl}(F) = \vec{\nabla} \times \vec{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & -x+y & -2z \end{vmatrix}$$

$$\hat{i}(0) - \hat{j}(0) + \hat{k}(-1-1) = -2\hat{k}$$

(b) $\vec{\nabla} \cdot \vec{G} = 0$

$\vec{\nabla} \times \vec{G}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 2x+3z & 3y \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(2-2) = 0$$

$\vec{\nabla} \times \vec{G} = 0$

$\Rightarrow \vec{G}$ is a gradient field

||| flashbacks

let $\vec{G} = \vec{\nabla}(\phi) = \mathrm{grad}(\phi)$

{ ||| flashbacks }

Let $\vec{G} = \vec{\nabla}(\Phi) = \text{grad}(\Phi)$

where $\Phi$ is a scalar function

$$\vec{G} = \nabla\vec{\Phi}$$

$$(2y)\hat{i} + \underline{(2x+3z)}\hat{j} + \underline{(3y)}\hat{k} = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$

$$\frac{\partial\Phi}{\partial x} = 2y$$

$$\Rightarrow \underline{\Phi(x,y,z) = 2yx + f(y,z)}$$

$$\frac{\partial\Phi(x,y,z)}{\partial y} = 2x + \frac{\partial f(y,z)}{\partial y} = \underline{2x+3z}$$

$$\Rightarrow \frac{\partial f(y,z)}{\partial y} = 3z$$

$$f(y,z) = 3yz + \underline{g(z)}$$

$$\Phi(x,y,z) = 2xy + 3yz + g(z)$$

$$\frac{\partial\Phi(x,y,z)}{\partial z} = \underline{3y + g'(z) = 3y}$$

$$g'(z) = 0$$

$$g(z) = c$$

$$\boxed{\Phi(x,y,z) = 2xy + 3yz + c}$$

$\vec{F}_e = \vec{G} = \nabla.V$

$(\underline{\vec{G}})$

$\Phi(x,y,z) = 2xy + 3yz + 1$

$2xy + 3yz + 2$

we observe that for different values of potential (a/c to our choice of c) we get same value of the field $\vec{G}$.

This freedom in choice of potential $\Phi$ is called

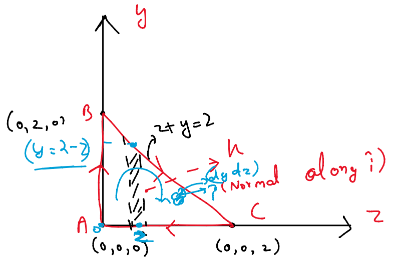
Gauge freedom

Potential

Gauge freedom

3.* Test the Stokes theorem for the vector $\vec{v} = xy\,\hat{i} + 2yz\,\hat{j} + 3z\,\hat{k}$ using a triangular area with vertices at (0,0,0), (0,2,0) and (0,0,2).

Sol$^n$




## Stokes Theorem

MAIN

$$\Rightarrow \iint (\vec{\nabla}\times\vec{v})\cdot d\vec{s} = \oint \vec{v}\cdot d\vec{l}$$

LHS RHS path integral (A → B → C → A)

$$\vec{V} = xy\,\hat{i} + 2yz\,\hat{j} + 3z\,\hat{k}$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3z \end{vmatrix}$$

$$= \hat{i}(0 - 2y) - \hat{j}(0 - 0) + \hat{k}(-x)$$

$$= -2y\,\hat{i} - x\,\hat{k}$$

$d\vec{s} = (dy\,dz\,\hat{i})$

$$\text{LHS} = \int_0^2\int_0^{2-z} (-2y\,\hat{i} - x\,\hat{k})\cdot(dy\,dz\,\hat{i}) = \int_0^2\int_0^{2-z} -2y\,dy\,dz = \int_0^2 -(2-z)^2\,dz$$

$$= \int_0^2 (4z - 4 - z^2)\,dz$$

$$= 2\times 4 - 4\times 2 - \frac{8}{3}$$

$$= -8/3$$

$\vec{l}\,(\vec{AB}) = 0\hat{i} + t\hat{j} + 0\hat{k}$

$d\vec{l}\,(\vec{AB}) = dt\,\hat{j}$

$x=0,\ y=t,\ z=0$ $\quad$ $x=0\ \ y=t,\ z=2-t$ $\quad$ $x=0, y=0,\ z=t$

$$\text{RHS} = \int \vec{v}\cdot d\vec{l}_{AB} + \int \vec{v}\cdot d\vec{l}_{BC} + \int \vec{v}\cdot d\vec{l}_{CA}$$

$$= \int_0^2 (xy\,\hat{i} + \cancel{2yz\,\hat{j}} + 3z\,\hat{k})\cdot(dt\,\hat{j}) + \int_2^0 (xy\,\hat{i} + 2yz\,\hat{j} + 3z\,\hat{k})\cdot(dt\,\hat{j} - dt\,\hat{k})$$

$x=0,\ y=t,\ z=0$

$$+ \int_2^0 (xy\,\hat{i} + 2yz\,\hat{j} + 3z\,\hat{k})\cdot(dt\,\hat{k})$$

$$= \int_2^0 [2t(2-t) - 3(2-t)]\cdot dt + \int_2^0 (3t\cdot dt)$$

$$= 5(-4) - \frac{2}{3}(-8) + 12$$

$$= \int_{2} \left[2 + (2-t) - 3(2-t)\right] . dt + \int_{2} (5t . 4t)$$

$$= \int_{2}^{0} (4t - 2t^2 - 6 + 6t) . dt = \int_{2}^{0} (10t - 2t^2 - 6) . dt = 5(-4) - \frac{2}{3}(-8) + 12$$

$$= -8 + \frac{16}{3} = \underline{-\frac{8}{3}} = \underline{LHS}$$

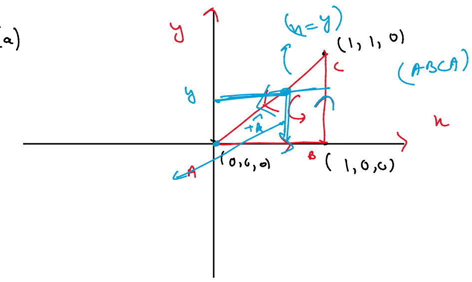
5.* A force defined by $\vec{F} = A(y^2\hat{i} + 2x^2\hat{j})$ is exerted on a particle which is initially at the origin of the co-ordinate system. A is a positive constant. We transport the particle on a triangular path defined by the points (0,0,0), (1,0,0), (1,1,0) in the counterclockwise direction.

(a) How much work does the force do when the particle travels around the path? Is this a conservative force? $\vec{F} = 0$ at origin

(b) The particle is placed at rest right at the origin. Is this a stable situation? Give any argument (mathematical, physical, intuitive) to justify the stability (or instability) of this situation.

Conservative force ↓ Work done along a closed path is zero

Sol^n (a)




$$W = \oint \vec{F} . \vec{dl} = \int (\vec{\nabla} \times \vec{F}) . \vec{ds}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2x^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(4x - 2y)$$

$$\vec{ds} = + dx . dy \, \hat{k}$$

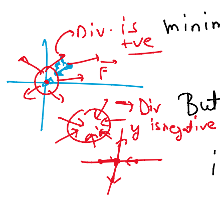
$$W = \int_0^1 \int_0^y \hat{k}(4x - 2y) . (+dx\,dy)\hat{k} = \int_0^1 \left[\int_y^1 (4x - 2y) . dx\right] . dy$$

$$\int_0^1 (2(1-y^2) - 2y(1-y)) . dy$$

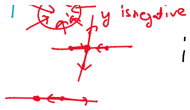
$$= \int_0^1 (2 - 2y) . dy = 2 - \frac{2 \times 1}{2} = 1$$

$\vec{F}$ is non conservative force

(b) A point of stability (instability) is a point of local minima (maxima) of potential $\phi$ corresponding to force $\vec{F}$. But for a non-conservative force we can't define a potential $\phi$. instead we use divergence of $\vec{F}$.




If $\vec{\nabla} \cdot \vec{F} < 0 \rightarrow$ stable eq^n

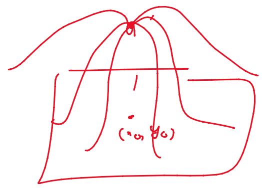



instead we use divergence of $\vec{F}$.

If $\vec{\nabla}\cdot\vec{F} < 0 \rightarrow$ stable eqn

↳ (for a conservative force $\vec{F}$ $\vec{\nabla}\cdot\vec{F} = \nabla^2\phi$)

Here $\vec{\nabla}\cdot\vec{F} = 0$ so the particle is not in stable eqn.




7.* Suppose that the height of a certain mountain (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 + 14x + 10y + 40),$$ HW0 2

where $x$ is the distance (in km) east, $y$ the distance north of the closest town.

(a) Where is the top of the mountain located, and how high is it?

(b) How steep is the slope (in feet per km) at a point 1 km north and 1 km east of the town? In what direction is the slope steepest, at that point?

Soln (a) We need to find maxima of function $h(x, y)$

$\nabla h = 0$ (at maxima)

$$\frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j} = 0$$

$$(2y - 6x + 14)\hat{i} + (2x - 8y + 10)\hat{j} = 0$$

$y - 3x + 7 = 0$ $\qquad$ $x - 4y + 5 = 0$

$-11y + 22 = 0$

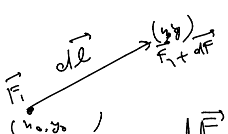
$y = 2$

$x = 3$

$h(3, 2) = 10(12 - 18 - 16 + 42 + 20 + 40) = 10(114 - 34) = 800$ ft.

(b) $\nabla h(1,1) = 10\hat{i} + 4\hat{j}$ $\qquad$ $\dfrac{2\hat{j} + 5\hat{i}}{\sqrt{29}}$

↳ also gives the direction of steepest slope




$$d\vec{F} = \frac{\partial \vec{F}}{\partial x}\cdot dx + \frac{\partial \vec{F}}{\partial y}\cdot dy + \frac{\partial F}{\partial z}\cdot dz$$

$$= \left(\frac{\partial \vec{F}}{\partial x}\hat{i} + \frac{\partial \vec{F}}{\partial y}\hat{j} + \frac{\partial \vec{F}}{\partial z}\hat{k}\right)(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

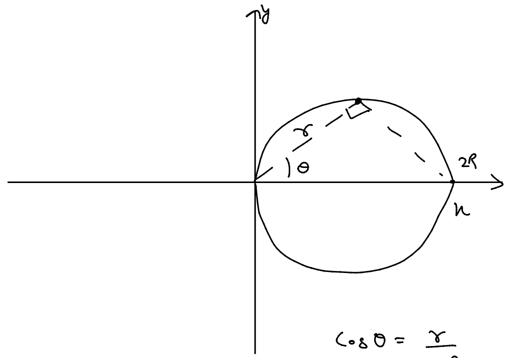
$$= (\nabla \vec{F})\cdot(d\vec{l})$$

$|d\vec{l}| \rightarrow$ fixed

$d\vec{l}$ is in direction of $(\nabla \vec{F})$

6. The area bounded by the curve $r = 2R\cos\theta$ has a surface charge density $\sigma(r,\theta) = \sigma_0 \frac{r}{R}\sin^4\theta$. What is the total amount of charge?

Sol$^n$




$\cos\theta = \frac{r}{2R}$

$r = 2R\cos\theta \longrightarrow$ a circle centred at $(R, 0)$

$x = r\cos\theta = 2R\cos^2\theta$

$y = r\sin\theta = 2R\cos\theta\sin\theta$

$\tan\theta = \frac{y}{x}$

$x = \frac{2R}{1 + y^2/x^2} \Rightarrow x^2 + y^2 = 2Rx$

$(x-R)^2 + y^2 = R^2$

$dq = \sigma\cdot(r\,dr\,d\theta)$

$q = \iint \sigma\cdot r\cdot dr\cdot d\theta = \iint \sigma_0\cdot\frac{r}{R}\sin^4\theta\cdot r\cdot dr\cdot d\theta$

for a given $\theta$ $r \in (0, 2R\cos\theta)$

$= \int_{-\pi/2}^{\pi/2}\int_0^{2R\cos\theta} \frac{\sigma_0 r^2}{R}\sin^4\theta\,dr\cdot d\theta$

$= \int_{-\pi/2}^{\pi/2} \frac{\sigma_0\sin^4\theta}{R}\,\frac{8R^3\cos^3\theta}{3}\cdot d\theta$

$= \frac{8\sigma_0 R^2}{3}\int_{-\pi/2}^{\pi/2} \sin^4\theta\cos^3\theta\cdot d\theta$

$= \frac{16\sigma_0 R^2}{3}\int_0^{\pi/2} \sin^4\theta\cdot\cos^3\theta\cdot d\theta$

Using red$^n$ formula

$$\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \begin{cases} \frac{[(m-1)(m-3)\cdots 1][(n-1)(n-3)\cdots 1]}{(m+n)(m+n-2)\cdots 2}\left(\frac{\pi}{2}\right) & m, n \text{ even} \\ \frac{[(m-1)(m-3)\cdots(2\text{ or }1)][(n-1)(n-3)\cdots(2\text{ or }1)]}{(m+n)(m+n-2)\cdots(2\text{ or }1)} & \text{otherwise} \end{cases}$$

$\Rightarrow q = \frac{16\sigma_0 R^2}{3}\times\frac{(\cancel{3}\times 1)(2)}{7\times 5\times\cancel{3}\times 1}$

$$\boxed{q = \frac{32}{105}\sigma_0 R^2}$$