Tutorial 10 Solutions

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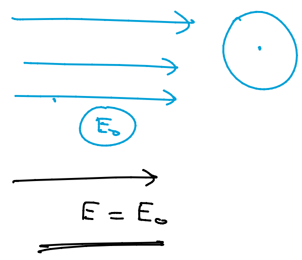
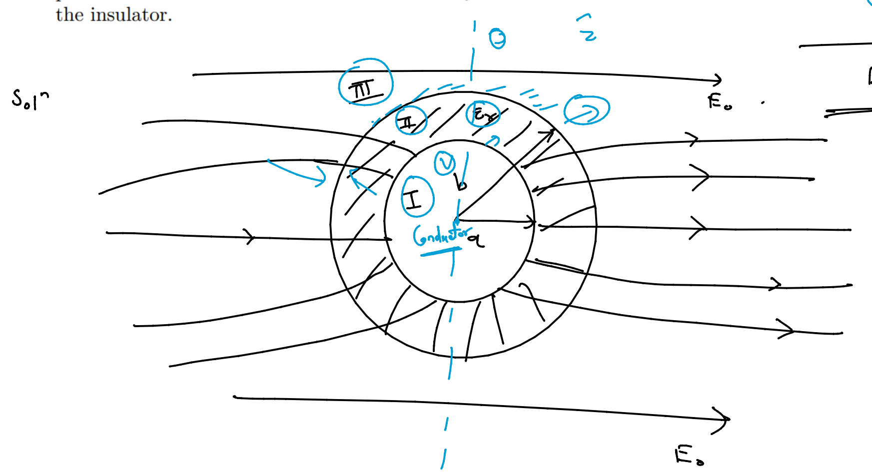


3. * An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b. This object is now placed in an otherwise uniform electric field E_0. Find the electric field in the insulator.

Sol^n



$$V_{II} = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$V_{III} = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$V_I = \underline{\text{const.}}$$

Boundary Conditions

(i) $\left.\dfrac{\partial V_{II}}{\partial\theta}\right|_{r=a} = 0$ Conductor

(ii) $\varepsilon_0 \left.\dfrac{\partial V_{III}}{\partial r}\right|_{r=b} = \varepsilon_0\varepsilon_r \left.\dfrac{\partial V_{II}}{\partial r}\right|_{r=a}$

(iv) $V_{III}|_{r=b} = V_{II}|_{r=b}$

$D_{\perp out} - D_{\perp in} = \sigma_f$

$\varepsilon_0 E_{\perp out} = \varepsilon_0\varepsilon_r E_{\perp in}$

$\cancel{\varepsilon_0}\dfrac{\partial V_{III}}{\partial r} = \cancel{\varepsilon_0}\varepsilon_r \dfrac{\partial V_{II}}{\partial r}$

(iii) $V_{III}(r\to\infty) \overset{?}{=} 0$ ✗

$E(r\to\infty) = E_0\hat{z}$

$-\nabla\cdot V = E_0\hat{z} \Rightarrow$

$\boxed{V_{III}(r\to\infty) = -E_0 z = -E_0 r\cos\theta}$

$\hookrightarrow \lim_{r\to\infty}\sum_{l=0}^{\infty}\left(A_l r^l + \dfrac{B_l}{r^{l+1}}\right) P_l(\cos\theta) = -E_0 r\cos\theta$

$$\sum_{l=0}^{\infty}(A_l r^l)(P_l(\cos\theta)) = -E_0 r\cos\theta$$

$$\boxed{\begin{aligned} &A_l = 0 \quad \forall\, l \neq 1 \\ &A_1 = -E_0 \end{aligned}}$$

(i) $\displaystyle\sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right) P_l'(\cos\theta)\Big|_{r=a} = 0 \quad \forall\,\theta$

For $l=0$ $P_l(\cos\theta) = 1 \Rightarrow P_l'(\cos\theta) = 0$

$\forall\, l\neq 0$ $a_l a^l + \dfrac{b_l}{a^{l+1}}$

$$\boxed{\frac{b_l}{a_l} = -a^{2l+1}}$$

(iv)

$\hookrightarrow \displaystyle\sum_{l=0}^{\infty}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right) P_l(\cos\theta) = \sum_{l=0}^{\infty}\left(a_l b^l + \frac{b_l}{b^{l+1}}\right) P_l(\cos\theta)$

For $l=0$ $\quad A_0 + \frac{B_0}{b} = a_0 + \frac{b_0}{b} \Rightarrow \boxed{B_0 = a_0 b + b_0}$

$(\because A_0 = 0)$

For $l=1$ $\quad A_1 b + \frac{B_1}{b^2} = a_1 b + \frac{b_1}{b^2} \Rightarrow A_1 b^3 + B_1 = a_1 b^3 + b_1$

$$\boxed{-E_0 b^3 + B_1 = a_1 (b^3 - a^3)}$$

For $l \neq 0, 1$ $\quad \frac{B_l}{b^{l+1}} = a_l b^l + \frac{b_l}{b^{l+1}} = a_l b^l - \frac{a_l a^{(2l+1)}}{b^{l+1}}$

$$\boxed{B_l = a_l \left(b^{2l+1} - a^{2l+1}\right)} \quad -(2)$$

(ii) $\quad \varepsilon_0 \sum_{l=0}^{\infty} \left[l A_l r^{l-1} - \frac{(l+1) B_l}{r^{l+2}} \right] P_l(\cos\theta) \Big|_{r=b}$

$$= \varepsilon_0 \varepsilon_r \sum_{l=0}^{\infty} \left[l a_l r^{l-1} - \frac{(l+1) b_l}{r^{l+2}} \right] P_l(\cos\theta) \Big|_{r=b}$$

For $l=0$ $\quad -\frac{B_0}{\cancel{b^2}} = \frac{-b_0 \varepsilon_r}{\cancel{b^2}}$

$$\boxed{B_0 = b_0 \varepsilon_r}$$

For $l=1$ $\quad A_1 - \frac{2B_1}{b^3} = \varepsilon_r \left(a_1 - \frac{2b_1}{b^3} \right)$

$$-E_0 - \frac{2B_1}{b^3} = \varepsilon_r \left(a_1 - \frac{2b_1}{b^3} \right)$$

$$-E_0 - \frac{2}{b^3} \left(a_1 (b^3 - a^3) + E_0 b^3 \right) = \varepsilon_r \left(a_1 + \frac{2 a_1 a^3}{b^3} \right)$$

$$-3E_0 - \frac{2a_1}{b^3}(b^3 - a^3) = \varepsilon_r a_1 \left(1 + \frac{2a^3}{b^3} \right)$$

$$-3E_0 = a_1 \left(\varepsilon_r \left(1 + \frac{2a^3}{b^3} \right) + 2 \left(1 - \frac{a^3}{b^3} \right) \right)$$

$$-3E_0 = a_1 \left((\varepsilon_r + 2) + \frac{2a^3}{b^3} (\varepsilon_r - 1) \right)$$

$$\boxed{a_1 = \frac{-3E_0}{(\varepsilon_r + 2) + \frac{2a^3}{b^3}(\varepsilon_r - 1)}}$$

$$\boxed{b_1 = \frac{+3E_0 a^3}{(\varepsilon_r + 2) + \frac{2a^3}{b^3}(\varepsilon_r - 1)}}$$

$$B_1 = a_1 (b^3 - a^3) + E_0 b^3$$

$$B_1 = a_1(b^3 - a^3) + E_0 b^3$$

$$B_1 = \frac{-3E_0(b^3 - a^3)}{(\varepsilon_r + 2) + \frac{2a^3}{b^3}(\varepsilon_r - 1)} + E_0 b^3$$

$$B_1 = E_0\left[\frac{-3(b^3 - a^3) + b^3(\varepsilon_r+2) + 2a^3(\varepsilon_r - 1)}{(\varepsilon_r + 2) + \frac{2a^3}{b^3}(\varepsilon_r - 1)}\right]$$

$$\boxed{B_1 = E_0\left[\frac{a^3(2\varepsilon_r + 1) + b^3(\varepsilon_r - 1)}{(\varepsilon_r + 2) + \frac{2a^3}{b^3}(\varepsilon_r - 1)}\right]}$$

$\underline{l \neq 0, 1}$

$$-\frac{(l+1)B_l}{b^{l+2}} = l a_l b^{l-1} - \frac{(l+1)b_l}{b^{l+2}}$$

$$-\frac{(l+1)B_l}{b^{l+2}} = l a_l b^{l-1} + \frac{(l+1)a_l a^{2l+1}}{b^{l+2}}$$

$$-(l+1)B_l = l a_l b^{2l+1} + (l+1)a_l a^{2l+1}$$

$$-(l+1)\left(a_l b^{2l+1} - a_l^{2l+1}\right) = l a_l b^{2l+1} + (l+1)a_l a^{2l+1}$$

$$a_l b^{2l+1}(2l+1) = 0 \Rightarrow a_l = 0$$
$$\Downarrow$$
$$\boxed{b_l, B_l = 0}$$

We've got expressions for A_1, B_1 & a_1, b_1 & two relations b/w a_0, b_0 & B_0. We need one more relation.

↳ We've not yet used the boundary condition

VI

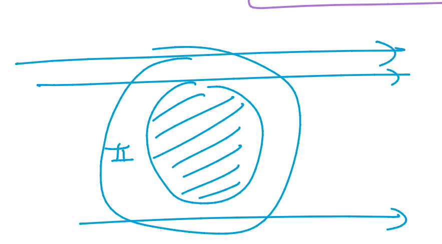
$$\varepsilon_0 \frac{\partial V_I}{\partial r} = 0 \qquad \varepsilon_0 \varepsilon_r \frac{\partial V_{II}}{\partial r}\bigg|_{r=a} = 0$$

Also $V_{II} = \left(a_0 + \frac{b_0}{r}\right) + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta$ $\quad (a_l, b_l = 0 \;\; \forall l \neq 0, 1)$

We'll only solve for $l = 0$ part $\because$ $l = 1$ is already sorted

$$\frac{\partial V_{II}}{\partial r} = 0 \Rightarrow -\frac{b_0}{r^2} = 0 \Rightarrow \boxed{b_0 = 0}$$

↳ $\boxed{B_0 = 0, A_0 = 0}$



Finally we get the potentials

$$V_{II}(r) = \frac{-3E_0}{\ldots} r\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

b^3

$$V_{\text{II}}(r) = \frac{-3E_0}{(\varepsilon_r+2)+\frac{2a^3}{b^3}(\varepsilon_r-1)}\, r\left(1-\frac{a^3}{r^3}\right)\cos\theta$$

$$V_{\text{III}}(r) = E_0 r\cos\theta - \frac{E_0 b^3\cos\theta}{r^2}\left[\frac{\frac{a^3}{b^3}(2\varepsilon_r+1)+(\varepsilon_r-1)}{(\varepsilon_r+2)+\frac{2a^3}{b^3}(\varepsilon_r-1)}\right]$$

$$\boxed{B_1 = E_0\left[\frac{a^3(2\varepsilon_r-5)-b^3(\varepsilon_r-1)}{(\varepsilon_r+2)+\frac{2a^3}{b^3}(\varepsilon_r-1)}\right]}$$

$P_n = \alpha_n E_x$

$\vec{E} = E_n\hat{i}$

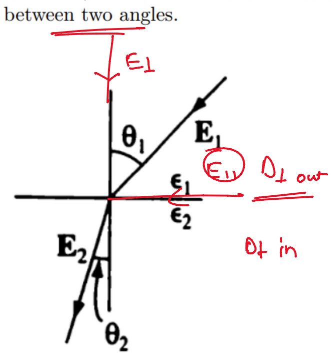
Non linear dielec.

$P_n = \alpha_{nn} E_n$

$\rightarrow P_y = \alpha_{yn} E_n$

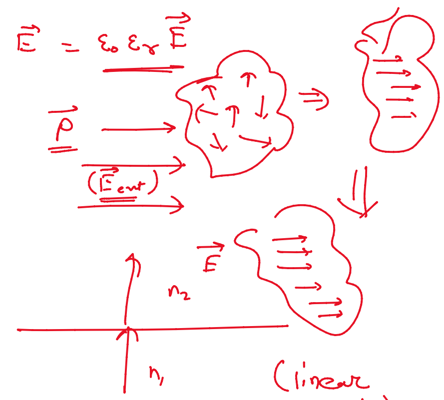
$\rightarrow P_z = \alpha_{zn} E_n$

4. * Two dielectrics having permittivity ϵ_1 and ϵ_2 have an interface which has no free charges. The electric field in medium 1 makes an angle θ1 with perpendicular of interface while, the field in medium 2 makes an angle θ2. Find the Relationship between two angles.



$\alpha\vec{E}$

$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = (\varepsilon_0+\alpha)\vec{E} = \varepsilon_0\varepsilon_r\vec{E}$



$D_{\perp out} - D_{\perp in} = \sigma_f$

$D_{\perp out} = D_{\perp in}$

$\varepsilon_1\varepsilon_0 E_{\perp 1} = \varepsilon_2\varepsilon_0 E_{\perp 2}$

$\varepsilon_1\varepsilon_0\frac{\partial V}{\partial n_1} = \varepsilon_0\varepsilon_2\frac{\partial V}{\partial n_2}$

$\boxed{\varepsilon_1\cancel{\varepsilon_0} E_1\cos\theta_1 = \varepsilon_2\cancel{\varepsilon_0} E_2\cos\theta_2}$

$E_1\sin\theta_1 = E_2\sin\theta_2$

$$\boxed{\frac{\varepsilon_1}{\varepsilon_2} = \frac{\tan\theta_1}{\tan\theta_2}}$$

Sol^n $\varepsilon_1\frac{\partial V}{\partial n_1} = \varepsilon_2\frac{\partial V}{\partial n_2}$

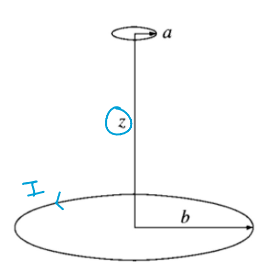
$D_{1\perp} = D_{2\perp}$

$\varepsilon_1 E_1\cos\theta_1 = \varepsilon_2 E_2\cos\theta_2$

also $E_\parallel$ is continuous

$E_1\sin\theta_1 = E_2\sin\theta_2$

7. * Two circular loops of wire share the same axis but are displaced vertically by a distance, z. The wire of radius a is considerably smaller than the wire of radius b.

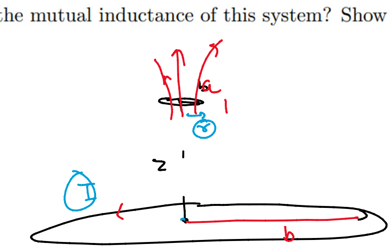


JEE (a) The larger loop (of radius b) carries a current I. What is the magnetic flux through the smaller loop due to the larger? (Hint: The field of the large loop may be considered constant in the region of the smaller loop.)

(b) If the same current I now flows in the smaller loop, then what is the magnetic flux through the larger loop? (Hint: The field of the smaller loop may be treated as a dipole.)

(c) What is the mutual inductance of this system? Show that $M_{12}=M_{21}$

Sol^n (a)



$\Phi = \int B\cdot da =$

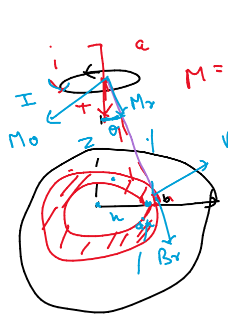
$\Phi_2 = \oint B_{21}\cdot da_1$

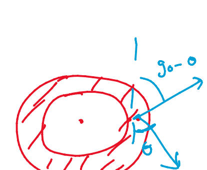
$B_{21} = \frac{\mu_0 I_2 b^2}{2(b^2+z^2)^{3/2}} \Rightarrow$

$\Phi_{21} = \oint B_{21}\cdot dA_2 = B_{21}\cdot A_2 = \frac{\mu_0 I_2 b^2}{2(b^2+z^2)^{3/2}}(\pi a^2)$

(b)

$M = (I\cdot\pi a^2)$





$B_\theta = \frac{\mu_0}{4\pi}\frac{M\sin\theta}{(n^2+z^2)^{3/2}}$

$B_r = \frac{\mu_0}{4\pi}\frac{2M\cos\theta}{(n^2+z^2)^{3/2}}$

$\sin\theta = \frac{n}{\sqrt{(n^2+z^2)}}$

$\cos\theta = \frac{z}{\sqrt{n^2+z^2}}$

$B_\perp = B_r\cos\theta - B_\theta\sin\theta$

$= \frac{\mu_0 M}{4\pi}\frac{(2\cos^2\theta-\sin^2\theta)}{(n^2+z^2)^{3/2}}$

$$B_\perp = \frac{\mu_0}{4\pi} M \frac{(2z^2 - r^2)}{(r^2+z^2)^{5/2}}$$

$$d\Phi_{12} = B_\perp \cdot dA = \frac{\mu_0}{4\pi} M \frac{(2z^2 - r^2)}{(r^2+z^2)^{5/2}} \cdot (2\pi r\, dr)$$

$$\Phi_{12} = \int_0^b \frac{\mu_0 M}{2} \frac{(2z^2 r - r^3)\, dr}{(r^2+z^2)^{5/2}}$$

$r = z\tan\theta$

$dr = z\sec^2\theta$

$\tan\alpha = b/z$

$$\phi = \int_0^\alpha \frac{\mu_0 M}{2} \cdot \frac{1}{z^2} \frac{(2\tan\theta - \tan^3\theta)\, z\sec^2\theta \, d\theta}{\sec^5\theta}$$

$$\phi = \int_0^\alpha \frac{\mu_0 M}{2z}\left(2\sin\theta\cos^2\theta - \sin^3\theta\right) d\theta$$

$$= \int \frac{\mu_0 M}{2z}\left(3\sin\theta\cos^2\theta - \sin\theta\right) d\theta$$

$$= \frac{\mu_0 M}{2z}\left. -\cos^3\theta + \cos\theta \right|_0^\alpha = \frac{\mu_0 M}{2z}(\cos\alpha - \cos^3\alpha)$$

$$= \frac{\mu_0 M}{2z}\left(\frac{z}{\sqrt{b^2+z^2}}\right)\left(1 - \frac{z^2}{b^2+z^2}\right)$$

$$\Phi_{12} = \frac{\mu_0 M}{2} \frac{b^2}{(z^2+b^2)^{3/2}} = \boxed{\frac{\mu_0 I_1 b^2 (\pi a^2)}{2(z^2+b^2)^{3/2}}}$$

(c) $\Phi_{12} = I_1 \times M_{12}$

Can prove $\boxed{M_{12} = M_{21}}$

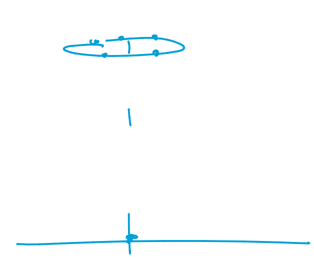
6. * A charge Q is distributed uniformly on a non-conducting ring of radius R and mass M. The ring is dropped from rest from a height h and falls to the ground through a non-uniform magnetic field $\mathbf{B}(\mathbf{r})$. The plane of the ring remains horizontal during its fall.

(a) Explain qualitatively why the ring rotates as it falls.

(b) Use Faraday's flux rule to show that the velocity of the center of mass of the ring when it hits the ground is

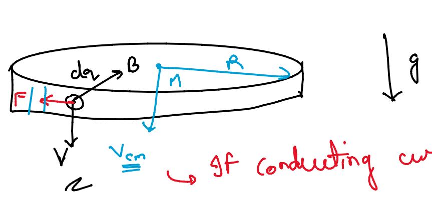
$$v_{CM} = \sqrt{2gh - \frac{Q^2R^2}{4M^2}[B_z(0) - B_z(h)]^2}$$

$\vec{B}(\vec{r})$



$\vec{E} = \frac{d\varepsilon}{dl}$

Sol^n (a)



↳ If conducting current starts to flow → as usually happens in EMI

→ If non-conducting ring starts to rotate → due to the force (and hence torque exerted)

(b) Emf induced $\varepsilon = -\frac{d\phi}{dt} = -\frac{d}{dt}(B_z \cdot \pi R^2) = -\pi R^2 \frac{dB_z}{dt}$

$\vec{E} = -\frac{d\varepsilon}{dl}$

$\varepsilon = -\oint E \cdot dl = -E \cdot 2\pi R \hat{\phi}$ (Radial Symm.)

$$E = \frac{-\mathcal{E}}{2\pi R}$$

$MR^2 \rightarrow$ ring

$$I\alpha = MR^2 \frac{d\omega}{dt} = \tau = (Q_0 E R) = \frac{Q_0 \mathcal{E}}{2\pi} = \frac{Q}{2\pi} \pi R^2 \int_0^z \frac{dB_z}{dt}$$

$$MR^2 \omega = \frac{Q_0 R^2}{2} \left(B_z(z) - B_z(0) \right)$$

$$W_{all} = Mgh = W_g = \Delta KE = \frac{1}{2} M V_z^2 + \frac{1}{2} M r^2 \omega^2 = \frac{1}{2} M V_z^2 + \frac{1}{2} \frac{Q_0 R^2}{4 M^2} \left(B_z(z) - B_z(0) \right)$$

$$\boxed{V_z = \sqrt{2gH - \frac{Q_0 R^2}{4M^2} \left(B_z(H) - B_z(0) \right)}}$$