Tutorial 11 Solution

21 June 2022 15:03

7. * A cylinder of radius R and infinite length is made of permanently polarized dielectric. The polarization vector $\vec{P}$ is proportional to radial vector $\vec{r}$ everywhere, $\vec{P}$=a$\vec{r}$ where a is positive constant. The cylinder rotates around its axis with an angular speed ω. This is a non-relativistic problem.

(a) Calculate electric field $\vec{E}$ at a radius r both inside and outside the cylinder.

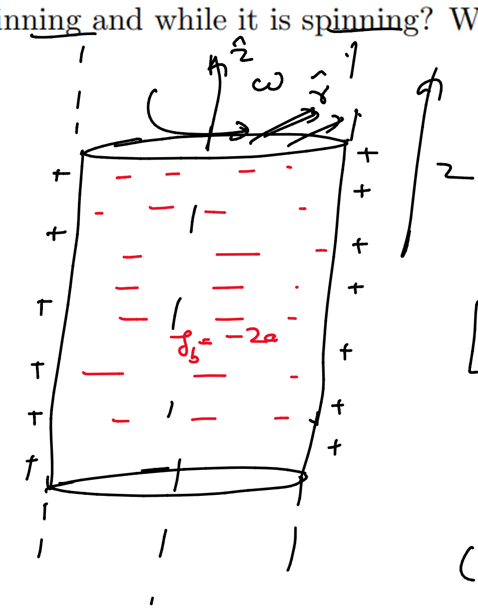
(b) Calculate magnetic field $\vec{B}$ at a radius r both inside and outside the cylinder.

(c) What is the total electromagnetic energy stored per unit length of the cylinder before it started spinning and while it is spinning? Where did the extra energy come from?

$\vec{r} = r\hat{r}$

Sol^n

$\sigma = aR$



$\vec{P} = a\vec{r} = ar\hat{r}$ ($\vec{r}$ is radial vector) → cylindrical coordinates

$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r}\frac{\partial}{\partial r}(ar^2) = -2a$

$\sigma_b(\text{curved surface}) = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = aR$

$\sigma_b(\text{circular surfaces}) = 0$ $(\hat{r}\cdot\hat{z} = 0)$

(a) $E(r<R) = \underbrace{E_\sigma}_{0} + E_\rho$

(σ is uniform)

$$\boxed{E_\rho \cdot 2\pi r l} = \frac{\overset{\text{const.}}{\rho}\cdot\pi r^2 l}{\varepsilon_0}$$

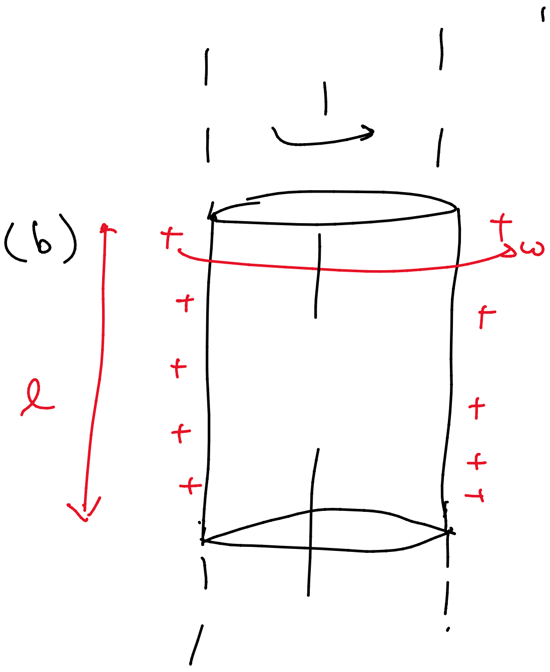
$$E_\rho = \frac{\rho r}{2\varepsilon_0} = \frac{-ar}{\varepsilon_0}$$

$$\boxed{E(r<R) = -ar/\varepsilon_0\,\hat{r}}$$

$E(r>R) = E_\sigma + E_\rho = 0$

Using gauss law & the fact that total charges induced σ_b & ρ_b are zero

(b)



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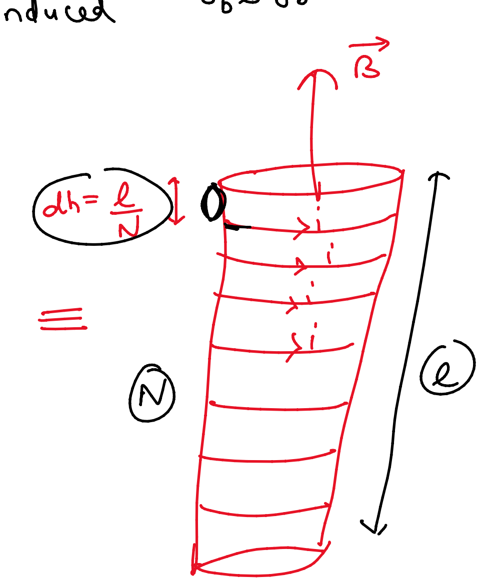


(The magnetic field due to rotating surface charge can be calculated by taking it to be equivalent to a solenoid)

$$\vec{B}_{solenoid}(r<R) = \mu_0 n I$$

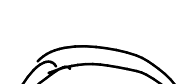
$$= \mu_0\left(\frac{N}{l}\right)\left(\frac{\sigma\cdot\cancel{2\pi} R\,dh}{\cancel{2\pi}/\omega}\right) \quad \frac{(dq)}{(2\pi/\omega)}$$

$$= \mu_0\left(\frac{N}{l}\right)\left(\sigma R\omega \times \frac{l}{N}\right) = \mu_0\sigma_b R\omega$$

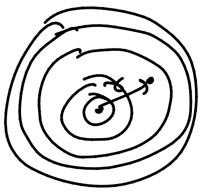
$\rho\cdot(2\pi r'\,dr')\,dh'$

$$\vec{B}_{solenoid}(r<R) = \mu_0\sigma R\omega\,\hat{z} = \mu_0 a R^2\omega\,\hat{z}$$

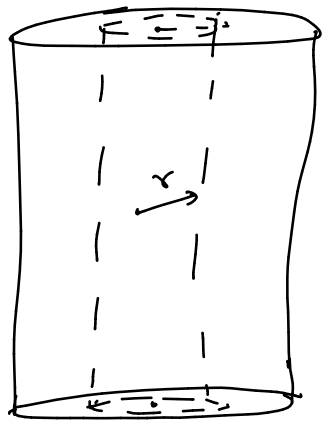
$$\vec{B}^{\sigma}(r>R) = 0$$



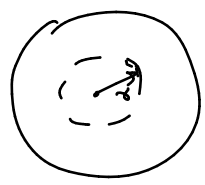
$\rho\cdot(2\pi r \ldots)$



$r' = r$ to R $\Rightarrow$ $\vec{B}^{\sigma}_{\text{Solenoid}}(r>R) = 0$



$\Rightarrow$ $\vec{B}^{\rho}_{\text{Solenoid}}(r<R) = \int_r^R dB^{\rho}_{\text{solenoid}} = \int_r^R \mu_0 \underbrace{\left(\frac{N}{\ell}\right)}_{n} \underbrace{\left(\rho\,\frac{2\pi r\,dr}{2\pi/\omega}\times\frac{\ell}{N}\right)}$



$r < r' < R$

Only solenoids with radius $> r$ would contribute

$$= \frac{\mu_0 \rho_s \omega}{2}(R^2-r^2) = -\underbrace{\mu_0 a\omega\,(R^2-r^2)}\,\hat{z}$$

$$\vec{B}^{\rho}_{\text{solenoid}}(r>R) = 0$$

$$\vec{B}_{\text{Total}}(r<R) = \vec{B}^{\sigma}(r<R) + \vec{B}^{\rho}(r<R) = \boxed{\mu_0 a\omega r^2\,\hat{z}}$$

$$\vec{B}_{\text{Total}}(r>R) = 0$$

(c) $E_{\text{Total}} = E_{\vec{E}} + E_{\vec{B}}$

Before spinning $\omega = 0 \Rightarrow \vec{B} = 0$

$$E_{\text{Total}}(\text{before spinning}) = \frac{1}{2}\varepsilon_0 \int_{\text{whole space}} E^2\,d\tau = \frac{1}{2}\varepsilon_0 \int \frac{a^2 r^2}{\varepsilon_0^2}(r\,dr\,d\theta\,dz)$$

$$= \frac{a^2}{2\varepsilon_0}\left(\frac{R^4}{4}\right)(2\pi)\,\ell$$

$$\boxed{E_{\text{Total}}(\text{before spinning}) = \frac{\pi a^2 R^4}{4\varepsilon_0}}$$ (Energy / unit length)

After spinning $E_{\vec{E}}$ remains unchanged, while $E_{\vec{B}}$ becomes

$$E_{\vec{B}} = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2\mu_0}\int \mu_0^2 a^2 \omega^2 r^4 \cdot (r\,dr)\,d\theta\,dz$$

$$E_{\vec{B}}(\text{per unit length}) = \frac{\mu_0}{\cancel{2}}\,a^2\omega^2\frac{R^6}{6}\times\cancel{2}\pi = \frac{\pi\mu_0 a^2\omega^2 R^6}{6}$$

$$\boxed{E_{\text{Total}}(\text{after spinning}) = \frac{\pi a^2 R^4}{4\varepsilon_0} + \frac{\pi\mu_0 a^2\omega^2 R^6}{6}}$$

Extra energy provided by the external agent rotating the cylinder.

8. Suppose,

$$E(\vec{r},t) = \frac{-1}{4\pi\epsilon_0}\frac{q}{r^2}\Theta(vt-r)\hat{r}, \qquad B(\vec{r},t) = 0$$

a) Show that they satisfy Maxwell's equations.
b) Determine ρ and $\vec{J}$
Here,

$$\Theta(x) = 1\,, \quad if\, x > 0$$
$$= 0\,, \quad if\, x \leq 0$$

and

$$\frac{d\Theta}{dx} = \delta(x)$$

Sol^n (a) $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$$\nabla\cdot\left(\frac{-q}{4\pi\epsilon_0}\left(\Theta(vt-r)\frac{\hat{r}}{r^2}\right)\right) = \frac{\rho}{\epsilon_0}$$

$$\frac{-q\,\Theta(vt-r)}{4\pi\epsilon_0}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) - \frac{q}{4\pi\epsilon_0}\nabla\cdot\left(\frac{\Theta(vt-r)\hat{r}}{r^2}\right)$$

$$\frac{-q\,\Theta(vt-r)\,4\pi\,\delta^3(\vec{r})}{4\pi\epsilon_0} - \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial\Theta(vt-r)}{\partial r}$$

$$-\frac{q\,\delta^3(\vec{r})}{\epsilon_0} + \frac{q}{4\pi\epsilon_0 r^2}\delta(vt-r) = \frac{\rho}{\epsilon_0}$$

$$\boxed{\rho = \frac{q\,\delta(vt-r)}{4\pi r^2} - q\,\delta^3(\vec{r})}$$

$\nabla\times\vec{E} =$

$\nabla\cdot\vec{B} = 0$

$$0 = \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = -\epsilon_0\frac{\partial}{\partial t}\left[\frac{-1}{4\pi\epsilon_0}\frac{q}{r^2}\Theta(vt-r)\hat{r}\right]$$

$$\boxed{\vec{J} = \frac{q}{4\pi r^2}\cdot v\cdot\delta(vt-r)\,\hat{r}}$$

5. * Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0 and frequency ω and phase angle zero, that is travelling (a) in the negative x direction and polarized in z direction, (b) traveling in the direction from the origin to the point (1,1,1) with polarization parallel to the xz plane.

$\hat{k} = -\hat{x}$

$\hat{n} = \hat{z}$

$\hat{k} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ $\qquad \hat{n} = a\hat{x} + b\hat{z}$

$\hat{n}\cdot\hat{k} = 0$

Sol: $\vec{k} \rightarrow$ wave vector

$|\vec{k}| \rightarrow$ wave number $= \dfrac{2\pi}{\lambda} = \dfrac{\omega}{c}$

$\hat{k} \rightarrow$ dir^n of propagation of wave

$\rightarrow \tilde{E}(\vec{r},t) = \tilde{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,\hat{n}$

$\hat{n} \rightarrow$ dir^n of polarization of $\vec{E}$

$\vec{r} \rightarrow x\hat{x} + y\hat{y} + z\hat{z}$

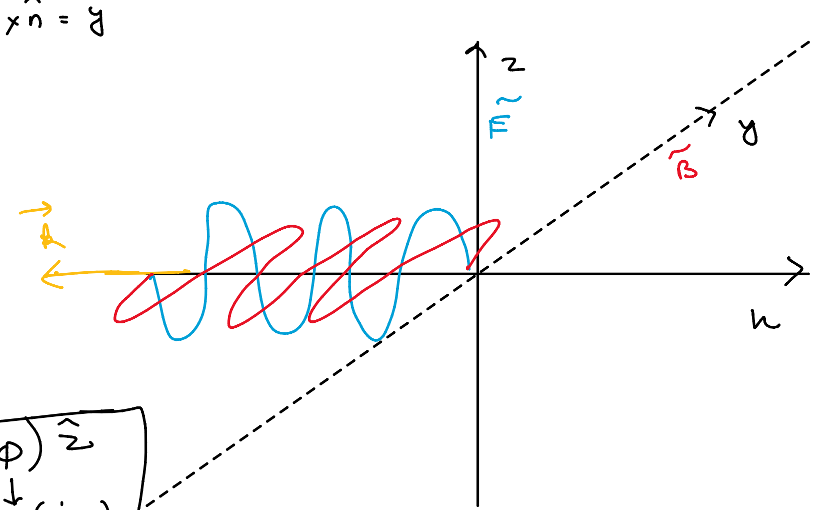
$\tilde{B}(\vec{r},t) = \tilde{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n})$

$\hat{n}\times\hat{k} \rightarrow$ dir^n of polarization of $\vec{B}$

(a) $\hat{k} = -\hat{x}$ $\qquad \hat{k}\times\hat{n} = \hat{y}$

$\hat{n} = \hat{z}$

$\vec{k}\cdot\vec{r} = -kx$



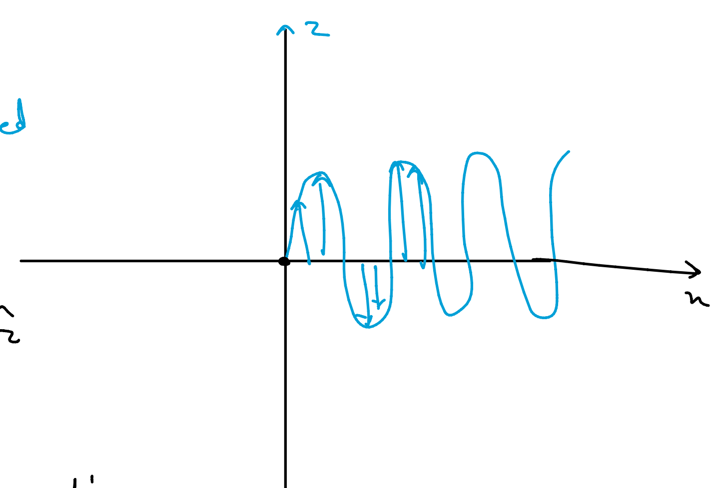
$\vec{E}(\vec{r},t) = E_0\, e^{i(-kx-\omega t)}\,\hat{z}$

$$\boxed{\vec{E}(\vec{r},t) = E_0 \cos(kx + \omega t + \underbrace{\phi}_{0\ (\text{given})})\,\hat{z}}$$

$\vec{B}(\vec{r},t) = B_0\, e^{i(-kx-\omega t)}\,\hat{y}$

$$\boxed{\vec{B}(\vec{r},t) = +B_0 \cos(kx+\omega t)\,\hat{y}} \rightarrow \boxed{\begin{array}{l} k = \frac{\omega}{c} \\ B_0 = E_0/c \end{array}}$$

z-polarized



(b) $\boxed{\hat{k} = (\hat{x}+\hat{y}+\hat{z})/\sqrt{3}}$ $\qquad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\hat{n} = a\hat{x} + b\hat{z}$

Since light is a transverse wave, dir^n of polarization is alway $\perp$ to wave vector $\Rightarrow \hat{k}\cdot\hat{n} = 0$

$a + b = 0$

$\Rightarrow \boxed{\hat{n} = \dfrac{\hat{x}-\hat{z}}{\sqrt{2}}}$ or $\dfrac{-\hat{x}+\hat{z}}{\sqrt{2}}$

i	j	k
1	0	-1
1	1	1

$$\boxed{\begin{array}{l} \vec{E}(\vec{r},t) = \pm E_0 \cos\left(\dfrac{k(x+y+z)}{\sqrt{3}} - \omega t\right)(\hat{x}-\hat{z})/\sqrt{2} \\ \vec{B}(\vec{r},t) = \pm B_0 \cos\left(\dfrac{k(x+y+z)}{\sqrt{3}} - \omega t\right)\underbrace{\left(\dfrac{\hat{x}-2\hat{y}+\hat{z}}{\sqrt{6}}\right)}_{(\hat{k}\times\hat{n})} \end{array}}$$

6. * Consider a propagating wave in free space given by $\rightarrow$ Grav. waves

$$\vec{E} = E_0 \frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{\sin(kr-\omega t)}{kr}\right]\hat{\phi} \qquad (1)$$

(a) Calculate the magnetic field $\vec{B}$ and the Poynting vector $\vec{S}$. You would need to use the expansions of $\nabla\times$ in spherical co-ordinates.

(b) What is the total average power radiated by the source? $\rightarrow$ Power through what surface ??

$\langle \vec{S} \rangle$

L.th Energy & Momentum

Power through ...

$\langle \vec{S} \rangle$

Sol^n EM waves carry both Energy & Momentum

Energy

energy density $\leftarrow U = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$

$$= \frac{1}{2}\epsilon_0 E^2 + \frac{(E^2)}{2\mu_0 c^2} = \frac{\epsilon_0 E^2}{2} + \frac{E^2}{2\mu_0 \cdot \frac{1}{\mu_0 \epsilon_0}} \Rightarrow \boxed{U = \epsilon_0 E^2}$$

Poynting Vector $\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})}$ $\hookrightarrow$ points in the dir^n of travelling wave

$\hat{E} = \hat{n}$

$\hat{B} = \hat{k}\times\hat{n}$

$\hat{E}\times\hat{B} = (\hat{n}\times(\hat{k}\times\hat{n}))$

$= (\hat{n}\cdot\hat{n})\hat{k} - \cancel{(\hat{n}\cdot\hat{k})}\hat{n}$

$= \hat{k}$

Energy (area - time)

(Energy flux density)

$\left(\frac{\text{Power}}{\text{Area}}\right)$

$$= \frac{1}{\mu_0}\frac{E^2}{c} = c\epsilon_0 E^2 = cU = \boxed{\vec{c}\,U}$$

$\frac{\langle P \rangle}{\text{Area}}$ (Time averaged)

Note that intensity is $\langle \vec{S} \rangle$

(Rwitaban Saviour!!)

(a) Given $\vec{E}$ find $\vec{B}$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = \frac{2E_0\cos\theta}{r^2}e^{i(kr-\omega t)}\left(1+\frac{i}{kr}\right)\hat{r}$$

$$-\frac{E_0\sin\theta}{r}e^{i(kr-\omega t)}(i)\left(k+\frac{i}{r}-\frac{1}{r^2}\right)\hat{\theta}$$

$$\vec{B} = -\left(\int(\nabla\times E)\cdot dt\right)$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

(b) $\langle\vec{S}\rangle = \sim = \frac{E^2\sin^2\theta}{2\mu_0 c r^2}\hat{r}$

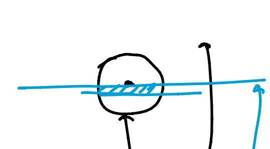
Avg. power / unit area

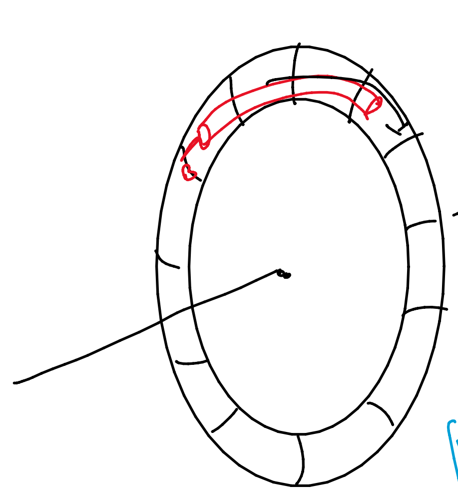
$$\langle P \rangle = \int \langle\vec{S}\rangle\cdot d\vec{S} = \int \frac{E^2\sin^2\theta}{2\mu_0 c r^2}\cdot \underbrace{r^2\sin\theta\, d\theta\, d\phi}$$

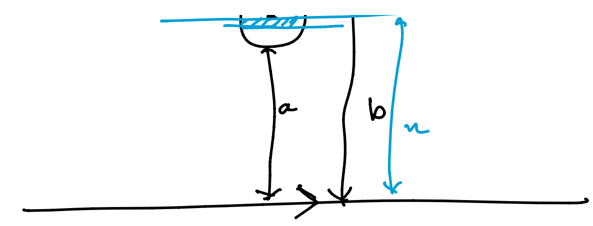
$\to$ (at a given radius r)

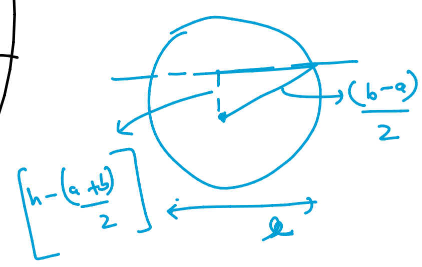
3. * An infinite wire carries a current up the rotational symmetry axis of a toroidal solenoid with N tightly wound turns and a circular cross section. The inner radius of the toroid is a and the outer radius is b. Find the mutual inductance M between the wire and the solenoid

Sol^n









$$d\phi = \frac{\mu_0 i}{2\pi r}\, l \cdot dr$$

$$\frac{l^2}{4} + \left[r - \frac{(a+b)}{2}\right]^2 = \frac{(b-a)^2}{4}$$

$$\frac{l^2}{4} = -ab - r^2 + r(a+b)$$

$$\underline{l = 2\sqrt{r(a+b) - r^2 - ab}}$$

$$d\phi_1 = \frac{\mu_0 i}{2\pi r}\, 2\sqrt{r(a+b) - r^2 - ab} \cdot dr$$

$$\phi_1 = \int_a^b \frac{\mu_0 i}{\pi r}\sqrt{r(a+b) - r^2 - ab} \cdot dr$$

$$I = \int_a^b \frac{\mu_0 i}{\pi r}\sqrt{(a-r)(r-b)} \cdot dr$$

$$\rightarrow r = a\cos^2\theta + b\sin^2\theta$$

$$dr = 2(b-a)\sin\theta\cos\theta$$

$$= \int_0^{\pi/2} \frac{\mu_0 i\,(a-b) \times 2(b-a) \times \sin^2\theta\cos^2\theta}{\pi\,(a\cos^2\theta + b\sin^2\theta)} \cdot d\theta$$

$$= -\frac{\mu_0 i\, 2\,(a-b)^2}{\pi}\int_0^{\pi/2} \frac{\sin^2\theta}{(a + b\tan^2\theta)} \cdot d\theta$$

$$\boxed{\phi_1 = \frac{\mu_0 i}{2}\left(\sqrt{b} - \sqrt{a}\right)^2}$$

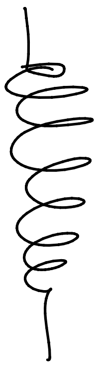
$$\phi_N = N\phi_1$$

1. * A very long solenoid of n turns per unit length carries a current which increases uniformly with time, $i = Kt$.

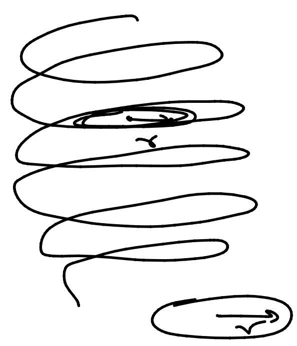
   (a) Calculate the electric field and magnetic field inside the solenoid at time t (neglect retardation).

   (b) Consider a cylinder of length l and radius equal to that of the solenoid, and coaxial with the solenoid. Find the rate at which energy flows into the volume enclosed by this cylinder and show that it is equal to $\frac{d}{dt}(\frac{1}{2}lLi^2)$, where L is the self-inductance per unit length of the solenoid.

Sol^n



$i = kt$



(a) $\vec{B} = \mu_0 n i = \mu_0 k t n \, \hat{z}$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 k \, \hat{z}$

$\int (\nabla \times \vec{E}) \cdot = \int_{\text{Surface}} -(\mu_0 k n)$

$\oint \vec{E} \cdot d\vec{l} = \mu_0 k n \pi r^2$

$E \cdot 2\pi r = \mu_0 k n \pi r^2$

$\boxed{E = \mu_0 k n r/2}$

(b)