Tutorial 3 Solution

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$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \vec{E} = 0$$

1. * Consider a vector field $\vec{F}(r)$, where $r = \vec{r}$ and $\vec{F}(r)$ dies faster than $\frac{1}{r}$ as $r \to \infty$, show the following results

   (a) Using Helmholtz theorem as discussed in Lecture 5, Show that $\vec{F}(r)$ may be written as

   $$\vec{F}(r) = -\nabla \frac{1}{4\pi}\int_V \frac{\nabla' \cdot \vec{F}(r')}{|r-r'|}d\tau' + \nabla \times \frac{1}{4\pi}\int_V \frac{\nabla' \times \vec{F}(r')}{|r-r'|}d\tau'$$

   (b) Derive the same expression for $\vec{F}(r)$ using

   $$\boxed{\vec{F}(r) = \int_V d\tau' \vec{F}(r')\delta^3(r-r')}$$

   boundary of the integral is to be understood at ∞.

   Hint: Use the following

   (i) $-4\pi\delta^3(r-r') = \nabla^2 \frac{1}{|r-r'|}$
   (ii) $\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2$
   (iii) $\nabla \frac{1}{|r-r'|} = -\nabla' \frac{1}{|r-r'|}$
   (iv) $\nabla \times \frac{\vec{F}(r')}{|r-r'|} = -\vec{F}(r') \times \nabla\left(\frac{1}{|r-r'|}\right)$ and 7(b) from Problem Set 2.

$$\frac{1}{|\vec{r}-\vec{r}'|}$$

$$\vec{F}(\vec{r}) = F_x(\vec{r})\hat{x} + F_y(\vec{r})\hat{y} + F_z(\vec{r})\hat{z}$$

$$\nabla \cdot \vec{F}(\vec{r}) = D(\vec{r})$$
$$\nabla \times \vec{F}(\vec{r}) = \vec{C}(\vec{r})$$

$D(\vec{r})$ & $\vec{C}(\vec{r})$ should fall off at a rate faster than $\frac{1}{r^2}$

Scalar

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) + \nabla \times \vec{A}(\vec{r})$$

$$\boxed{U(\vec{r}) = \frac{1}{4\pi}\int_V \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'}$$

Position of the point where field is required

location of source

$$\boxed{\vec{A}(\vec{r}) = \frac{1}{4\pi}\int_V \frac{\vec{C}(\vec{r}')\,d\tau'}{|\vec{r}-\vec{r}'|}}$$

Sol^n

(b) $$\vec{F}(r) = \int_V d\tau' \vec{F}(r') \cdot \left(-\frac{1}{4\pi}\right)\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{1}{4\pi}\nabla^2 \int_V d\tau' \frac{\vec{F}(\vec{r}')}{|r-r'|}$$

$$= \frac{1}{4\pi}\left[\nabla \times \nabla \times \int_V d\tau' \frac{\vec{F}(\vec{r}')}{|r-r'|} - \nabla\left[\nabla \cdot \int_V d\tau' \frac{\vec{F}(\vec{r}')}{|r-r'|}\right]\right]$$

$$\nabla \cdot \vec{F}(\vec{r}')$$

$$\nabla \cdot (f\vec{v}) = (\nabla f)\cdot\vec{v} + f(\nabla \cdot \vec{v})$$

$$(\nabla \times (f\vec{v}) = f(\nabla \times \vec{v}) - (\nabla f)\times\vec{v}$$

$$= \frac{1}{4\pi}\left[-\nabla \times \int_V d\tau' \left\{\nabla\left(\frac{1}{|r-r'|}\right) \times \vec{F}(\vec{r}')\right\} - \nabla \int_V \vec{F}(\vec{r}')\cdot\nabla\left(\frac{1}{|r-r'|}\right)\right]$$

$$-\nabla' \frac{1}{|\vec{r}-\vec{r}'|} \times \vec{F}(\vec{r}')$$

$$= \frac{1}{4\pi}\left[-\nabla \times \int d\tau' \left[\vec{F}(\vec{r}') \times \nabla' \frac{1}{|r-r'|}\right] + \nabla \int \left[\vec{F}(\vec{r}')\cdot\nabla' \frac{1}{|r-r'|}\right]\right]$$

$$\nabla \times (f\vec{v}) = f(\nabla \times \vec{v}) - (\nabla f)\times\vec{v}$$

$$= \frac{1}{4\pi}\left[-\nabla \times \int d\tau' \left[\nabla \times \left(\frac{\vec{F}(\vec{r}')}{|r-r'|}\right) - \frac{\nabla \times \vec{F}(\vec{r}')}{|r-r'|}\right] + \nabla \int \nabla' \cdot \left(\frac{\vec{F}(\vec{r}')}{|r-r'|}\right)\right.$$

$$-\nabla' \cdot \vec{F}(\vec{r}')\,d\tau$$

$$= \frac{1}{4\pi}\left[-\nabla \times \int d\tau \; \ldots \frac{}{|r-r'|} \ldots \right] \qquad - \frac{\nabla' \cdot \vec{F}(\vec{r}')}{|r-r'|} \, d\tau$$

$$= \frac{1}{4\pi}\left[-\nabla \times \oint\!\!\oint \frac{\vec{F}(\vec{r}')}{|r-r'|} \times d\vec{s} + \nabla \times \int \frac{\nabla \times \vec{F}(\vec{r}')}{|r-r'|} \right]$$

$n > 1$

$$= \left[\frac{1}{|\vec{r}-\vec{r}'|} \right] + \frac{1}{4\pi}(\nabla)\left[\oint\!\!\oint \frac{\vec{F}(\vec{r}') \cdot (d\vec{s}')}{|r-r'|} - \int \frac{\nabla' \cdot \vec{F}(\vec{r}')}{|r-r'|} \, d\tau' \right]$$

$\left(\frac{1}{r'}\right)^n$ $\quad (r'^2)$ $\quad \frac{1}{r'} \cdot (r')^2$

$\nabla f =$ $\qquad R = \infty$

$$\iint \left(\frac{1}{r'}\right) = 0 \qquad r' = \infty$$

$$\iint_{r' = \infty} \vec{F}(\vec{r}) \cdot \frac{1}{|\vec{r}-\vec{r}'|} \cdot (d\vec{s}')$$

$$\nabla \iint \frac{\vec{F}(\vec{r}')(d\vec{s}')}{|\vec{r}-\vec{r}'|} \qquad \frac{1}{r'}\left(\;\right) \left(\frac{1}{r'}\right) \cdot (r'^2 \sin\theta)$$

$$\oint \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|^2} \; \hat{\mathscr{r}} \; (d\vec{s}') \qquad \propto (r')^2 \qquad - \oint \frac{(c)}{(r')}$$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ $\quad \propto (r')^2$

Soln

$$\lim_{r' \to \infty} \nabla \left[\iint \frac{\vec{F}(\vec{r}') \cdot (d\vec{s}')}{|\vec{r}-\vec{r}'|} \right] \qquad (\nabla) \iint \frac{\vec{F}(\vec{r}') \, r'^2 \sin\theta' \, d\theta' \, d\phi'}{|\vec{r}-\vec{r}'| \to r'}$$

$$\nabla \iint \vec{F}(\vec{r}') \cdot r'^2 \qquad \nabla \left[\frac{F(\vec{r}')}{r'} \right]$$

$$\frac{\vec{F}(\vec{r}')(\;)}{|\vec{r}-\vec{r}'|^2} \; \hat{\mathscr{r}}$$

4. Evaluate the following integral

$$\int_V \mathbf{r} \cdot (\mathbf{d} - \mathbf{r})\delta^3(\mathbf{e} - \mathbf{r})d\tau$$

where $\mathbf{d} = (5, 5, 5)$, $\mathbf{e} = (15, 19, 17)$, and V is a sphere of radius 7 centered at (10, 15, 19).

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7. * After an extremely precise measurement, it was revealed that the actual force between two point charges is given by -

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}\hat{\mathbf{r}}$$

Where λ is a constant with dimensions of length, and it is a huge number which is why the correction is tiny and difficult to notice.
Does this electric field results from a scalar potential? Justify.
And if yes, find the potential due to a point charge q placed at the origin using infinity as your reference.

Sol^n $\quad \nabla \times \left(\underbrace{\frac{q_2}{4\pi\varepsilon_0 r^2}\left(1+\frac{r}{h}\right)e^{-r/h}\cdot\hat{r}}_{\text{Field in radial}} \right) = 0$

dir^n $\longrightarrow$ generated from a scalar pot.

$$\nabla\psi = (\quad)$$

$$\frac{\partial\psi}{\partial r} = \frac{q_2}{4\pi\varepsilon_0 r^2}\left(1+\frac{r}{h}\right)e^{-r/h} \qquad \frac{\partial\psi}{\partial\theta} = 0 \qquad \frac{\partial\psi}{\partial\phi} = 0$$

(2) $\quad \underbrace{\delta[g(x)] = \sum_m \frac{1}{|g'(x_m)|}\delta(x-x_m)}$

$$\int_{-\infty}^{\infty} f(x)\cdot\delta[g(x)] = \sum_m \int_{x_m-\varepsilon_0}^{x_m+\varepsilon_0} f(x)\cdot\delta[g(x)]$$

$g(x_m)=0 \quad \forall$

$x_1, x_2 \;\ldots\; x_m$

$g(x_i)=0 \quad \forall\, i\in\{1,2,\ldots m\}$

$$= \sum_m \int_{x_m-\varepsilon_0}^{x_m+\varepsilon_0} f(x)\cdot\delta\left[\cancel{g(x_m)}^{0} + g'(x_m)\cdot(x-x_m) + \underbrace{\frac{g''(x_m)}{2}(x-x_m)^2 + \ldots}\right]$$

$\delta(ax) = \frac{x}{|a|}$

$$= \sum_m \int_{x_m-\varepsilon_0}^{x_m+\varepsilon_0} f(x)\cdot\delta\left[g'(x_m)(x-x_m)\right]$$

$$= \sum_m \int_{x_m-\varepsilon_0}^{x_m+\varepsilon_0} f(x)\cdot\frac{\delta(x-x_m)}{|g'(x_m)|}$$

$\int_{-\infty}^{\infty} f(x)\cdot\delta(x-x_m)$

$$= \frac{1}{|g'(x_n)|}\sum_m \int_{x_m-\varepsilon_0}^{x_m+\varepsilon_0} f(x)\cdot\delta(x-x_m)$$

$$\sum_m \frac{1}{|g'(x_m)|}\int f(x)\cdot\delta(x-x_m)$$

$$\int_{-\infty}^{\infty} f(x)\cdot\delta g(x) =$$

$$= \int f(x)\left[\sum_m \frac{\delta(x-x_m)}{|g'(x_m)|}\right]$$

$$\boxed{\delta\left[g(x)\right] = \sum_m \frac{\delta(x-x_m)}{|g'(x_m)|}}$$