Tutorial 5 Solution

30 April 2022 11:37

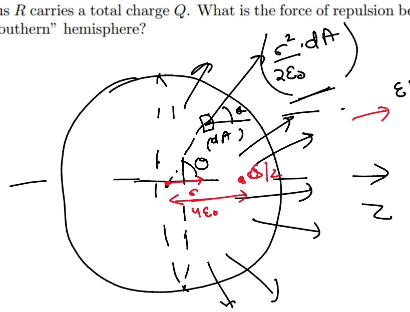
9.* (a) Find the average potential over a spherical surface of radius R due to a point charge q located inside. Show that, in general,

$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$$

where V_{center} is the potential at the center due to all the external charges and Q_{enc} is the total enclosed charge.
(b) Find the general solution to Laplace's equation in spherical coordinates for the case where V depends only on r. Do the same for cylindrical coordinates assuming V depends only on s.

3. A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

Sol^n



Electrostatic Pressure $\frac{\sigma^2}{2\epsilon_0}$

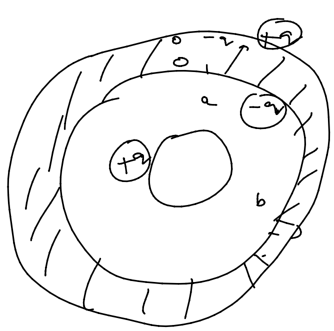
$$dF = \iint_{\text{hemisphere}} \frac{\sigma^2}{2\epsilon_0} \cdot dA \cos\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\frac{Q}{4\pi R^2}\right)^2 \cdot \frac{1}{2\epsilon_0} \left(R^2 \sin\theta \, d\theta \, d\phi\right) \cos\theta$$

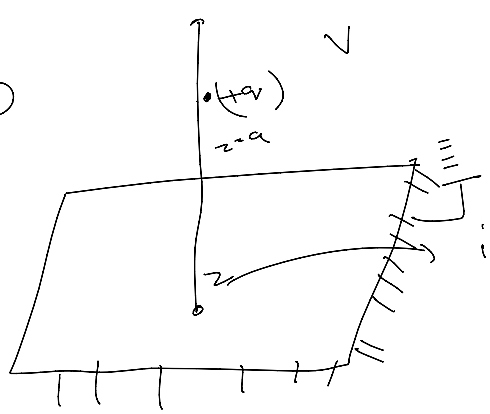
$$= \frac{Q^2}{16\pi^{\cancel{2}} \epsilon_0 \cdot \cancel{2}} \times \frac{\cancel{2\pi}}{R^2} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$= \frac{Q^2}{16\pi\epsilon_0 R^2} \times \frac{1}{4} \times 2 = \frac{Q^2}{32\pi\epsilon_0 R^2}$$

(5)



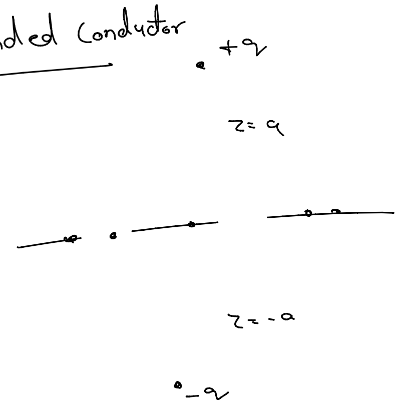
(6)



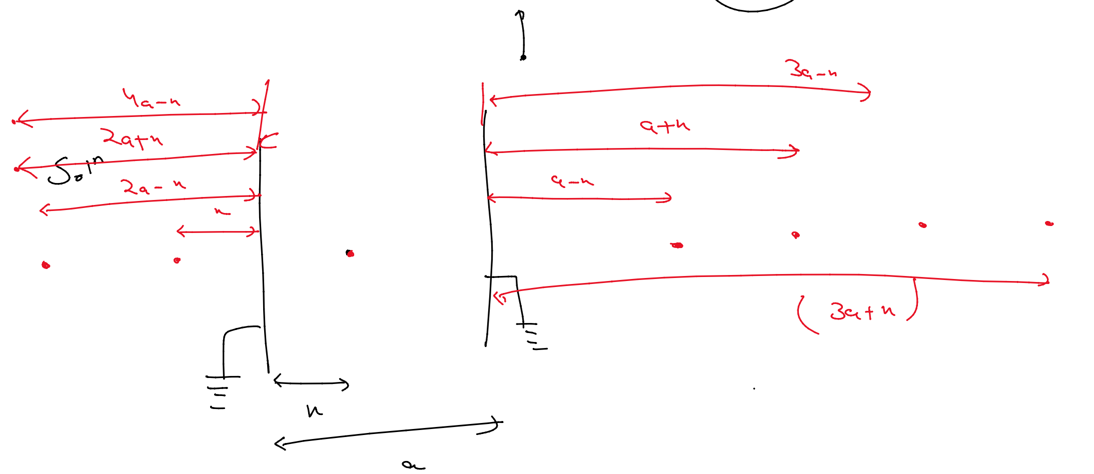
$V(z=0) = 0 \rightarrow$ grounded Conductor

$V(z \to \infty) = 0$

infinite conductor



$$2\pi\sqrt{\frac{GM}{a}}$$



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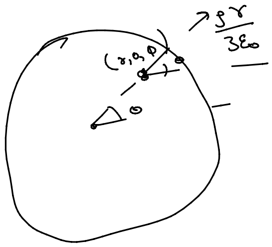

$$dE_z = \frac{\left(\frac{\rho r}{3\varepsilon_0}\cdot\cos\theta\right)\cdot\left(\rho r^2 \sin\theta\, dr\, d\theta\, d\phi\right)}{1}$$

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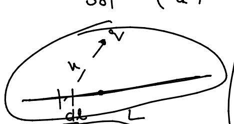
$$\frac{KQ_{enc}}{R}$$

$$V_{ave} = V_{ext,\,center} + V_{int,\,center}$$

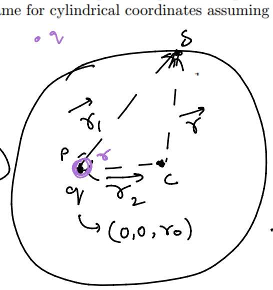
$$(V_{ave},\, q,\, \text{Line-}L) = \frac{\left(\frac{Kq}{h}\cdot dl_1 + \frac{Kq}{h}\cdot dl_2 + \cdots\right)}{L}$$

q, h, P, L

Sol^n (a)



$$\frac{\int \frac{Kq\,dl}{h}}{L}$$



$$(V_{avg})_{inside} = \frac{\iint \frac{Kq}{|\vec{r}_1|}\, dA}{4\pi R^2} = \frac{\iint \frac{Kq\cdot R^2 \sin\theta\, d\theta\, d\phi}{|\vec{r}-\vec{r}_2|}}{4\pi R^2}$$

$$\vec{CS} = \vec{PS} + \vec{CP}$$

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

(r_0, θ_0, ϕ_0)

$(0, 0, r_0)$

For simplicity we assume the charge to be located on z-axis.

$$\begin{matrix} r_0 \sin\theta_0 \cos\phi_0 \\ r_0 \sin\theta_0 \sin\phi_0 \\ r_0 \cos\theta_0 \end{matrix}$$

$$= \frac{\iint \frac{Kq\cdot \sin\theta\, d\theta\, d\phi}{\sqrt{(R\sin\theta\cos\phi - 0)^2 + (R\sin\theta\sin\phi - 0)^2 + (R\cos\theta - r_0)^2}}}{4\pi}$$

$$= \frac{1}{4\pi}\iint \frac{Kq \sin\theta\, d\theta\, d\phi}{\sqrt{(R^2+r_0^2) - 2Rr_0\cos\theta}}$$

$$= \frac{1}{4\pi}\iint \sqrt{(R^2+r_0^2) - 2Rr_0\cos\theta}$$

$\theta \to 0, \pi$

Substitute $R^2 + r_0^2 - 2R\,r_0\cos\theta = t$

$+2Rr_0 \sin\theta\, d\theta = dt$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{(R-r_0)^2}^{(R+r_0)^2} \frac{Kq}{2Rr_0}\frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{Kq}{Rr_0}\left(\sqrt{(R+r_0)^2} - \sqrt{(R-r_0)^2}\right)$$

$$= \boxed{\frac{Kq}{R}}$$

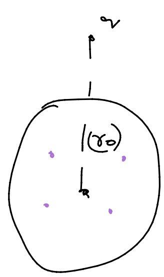
avg. potential due to inside charge

$(R+r_0) - (r_0-R)$

$\frac{Kq}{2Rr_0}\cdot 2R = \frac{Kq}{r_0}$

For case $r_0 > R$ (avg. pot. due to outside charges)

Avg. pot. due to outside charge → ??



$$V_{avg} = \frac{K(q_1+q_2+ \dots q_n)}{R} = \frac{KQ_{enc}}{R} + \frac{KQ_{outside}}{R_0}$$

in

(b) $\nabla^2 V(r) = 0$

3D Laplacian (Spherical)

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \cancel{\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)}^{0} + \cancel{\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2}}^{0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r)}{\partial r}\right) = 0$$

$$r^2\frac{\partial V(r)}{\partial r} = C$$

$$\frac{\partial}{\partial r}V(r) = \frac{C}{r^2}$$

$$V(r) = \frac{-C}{r} = \frac{k}{r}$$

$\nabla^2 V(s) = 0$

3 D Laplacian (Cylindrical)

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{1}{s}\frac{\partial V(s)}{\partial s} + \frac{\partial^2}{\partial s^2}V(s) = 0$$

$$\frac{\partial}{\partial s}V(s) + s\frac{\partial^2}{\partial s^2}V(s) = 0$$

$$\frac{\partial}{\partial s}\left[s\frac{\partial V(s)}{\partial s}\right] = 0$$

$$s\frac{\partial V(s)}{\partial s} = k$$

$$\frac{\partial V(s)}{\partial s} = \frac{k}{s}$$

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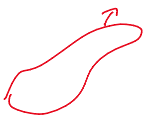
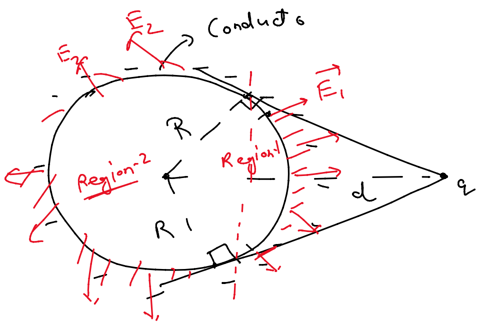
10.* A point charge $+q$ is placed at a distance d from the centre of a conducting sphere of radius R $(d > R)$ Show that if the sphere is grounded, the ratio of the charge on the part of the sphere visible from $+q$ to that on the rest is $\sqrt{\frac{d+R}{d-R}}$.

Sol^n →



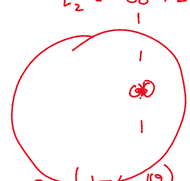


$$q_1 = \int \sigma_1 \cdot ds \qquad q_2 = \int \sigma_2 \cdot ds$$

$$q_1 = \underbrace{\int \varepsilon_0 \vec{E}_1 \cdot d\vec{S}_1}_{} \qquad q_2 = \underbrace{\int \varepsilon_0 \vec{E}_2 \cdot d\vec{s}_2}_{}$$

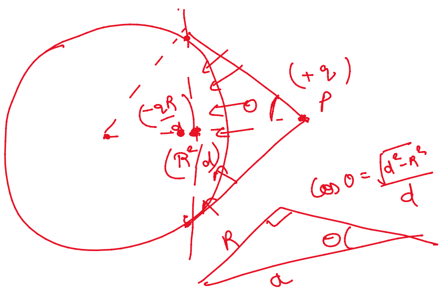
$$= \varepsilon_0 \Phi_1 \qquad q_2 = \varepsilon_0 \Phi_2$$

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$



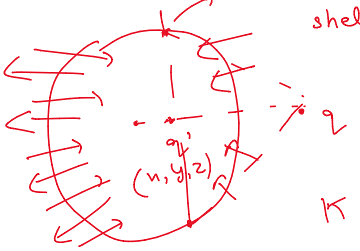
→ mag. of induced charge → $\frac{-qR}{d}$

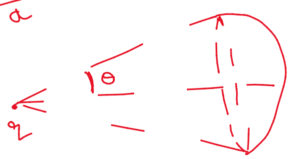
→ $\frac{R^2}{d}$



$$\cos\theta = \frac{\sqrt{d^2-R^2}}{d}$$

$$\frac{q_1}{q_2} = \frac{\Phi_1}{\Phi_2} = \left(\frac{-qR}{d(2\varepsilon_0)}\right) - \frac{q}{2\varepsilon_0}(1-\cos\theta)$$





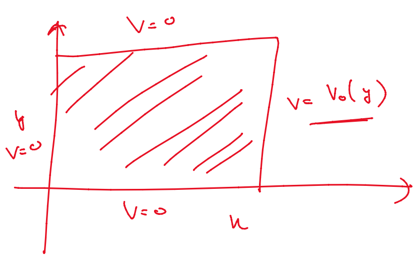
$$\text{flux} = \frac{q}{2\varepsilon_0}(1-\cos\theta) \qquad (\theta = \pi) \qquad \frac{q}{\varepsilon_0}$$

$$= \frac{q}{\varepsilon_0}\,\frac{2\pi(1-\cos\theta)}{4\pi}$$

$$\frac{q_1}{q_2} = \frac{\Phi_1}{\Phi_2} = \frac{\left(-\frac{qR}{d(2\varepsilon_0)}\right) - \underbrace{\frac{q}{2\varepsilon_0}\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)}_{}}{\left(-\frac{qR}{d(2\varepsilon_0)}\right) + \frac{q}{2\varepsilon_0}\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)} \qquad = \frac{q}{2\varepsilon_0}(1-\cos\theta)$$

12.* A rectangular pipe running parallel to the z-axis (from $-\infty$ to $+\infty$) has three grounded metal sides at $y = 0, y = a$ and $x = 0$. The fourth side at $x = b$ is maintained at a specified potential $V_0(y)$.
(a) Develop a general formula for the potential within the pipe.
(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

Sol^n



(a) $\nabla^2 V(x, y) = 0$

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

$$V(x, y) = f(x) \cdot g(y)$$

$$g(y)\,\frac{d^2 f^2(x)}{d x^2} + f(x)\,\frac{d^2 g^2(y)}{d y^2} = 0$$

$V(x=0, y) = 0$

$V(x, y=0) = 0$

$V(x, y=a) = 0$

$V(x=b, y) = V_0(y)$

$$\underbrace{\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2}} + \underbrace{\frac{1}{g(y)}\frac{d^2 g(y)}{dy^2}} = 0$$

$$\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2} = k^2 \qquad \frac{1}{g(y)}\frac{d^2 g(y)}{dy^2} = -k^2$$

$$\frac{d^2 f(x)}{dx^2} = k^2 f(x) \qquad \frac{d^2 g(y)}{dy^2} = -k^2 g(y)$$

$$\downarrow$$

$$f(x) = Ae^{kx} + Be^{-kx} \qquad g(y) = C\sin ky + D\cos ky$$

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

$$V(x=0, y) = (A+B)(\underbrace{C\sin ky + D\cos ky}) = 0 \quad \forall y$$

$$A + B = 0$$

$$V(x,y) = A(e^{kx} - e^{-kx})(C\sin ky + D\cos ky)$$

$$V(x, y=0) = A(e^{kx} - e^{-kx})(D) = 0 \quad \forall x$$

$$D = 0$$

$$V(x,y) = A(e^{kx} - e^{-kx})(\sin ky) = A\sinh kx \cdot \sin ky$$

$$V(x, y=a) = A(e^{kx} - e^{-kx})(\sin ka) = 0 \quad \forall x$$

$$ka = n\pi$$

$$\boxed{k = n\pi/a}$$

$$V(x,y) = A\sinh kx \cdot \sin ky \qquad \boxed{k = \frac{n\pi}{a}}$$

$$V(x=b, y) = V_0(y) = A\sinh kb\,\sin ky$$

$$V(x,y) = \sum_{n=0}^{\infty} C_n \sinh\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}\right)y$$

$$V_0(y) = V(b,y) = \sum^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi}{a}\right)y$$

$$\cancel{A\sin kx\,(\underbrace{e^{ky} - e^{-ky}})}$$

$$V(x, y=a) = A\sin kx\,(e^{ka} - e^{-ka}) = 0 \quad \forall x$$

$$\boxed{k = 0}$$

$$\boxed{V(x,y) = 0} \rightarrow \text{Trivial sol}^n$$

$$V_0(y) = V(b, y) = \sum_{n=0}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a}\right) y$$

$$\int_0^a V_0(y) \cdot \sin\left(\frac{m\pi}{a}\right) y \, dy = \int_0^a \sum_{n=0}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a}\right) y \sin\left(\frac{m\pi}{a}\right) y \cdot dy$$

if $V_0 \to \underline{\text{const.}}$

$$\underbrace{\int_0^a V_0(y) \sin\left(\frac{n\pi}{a}\right) y \cdot dy = C_n \sinh\left(\frac{n\pi b}{a}\right) \left(\;\right)}$$

$$\sum_{n=1,3,5\ldots} \frac{\frac{2a}{n\pi} \sinh\left(\frac{n\pi y}{a}\right) \left(\sin \frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

$$\frac{1}{n\pi/a}\left(1 - \cos n\pi\right) = \frac{2a}{n\pi} \quad \text{if } n \to \text{odd}$$

$$0 \quad \text{if } n \to \text{even}$$