Tutorial 6 Solution

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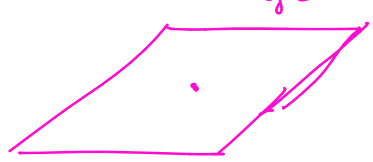
2. * A cubical box of side length a consists of five metal plates welded together and grounded. The sixth plate at the top is insulated from the rest and maintained at V_0.

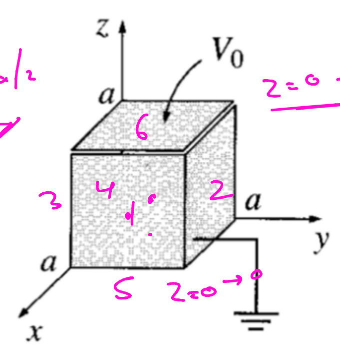
(a) Argue that the potential at the centre should be $\frac{V_0}{6}$

(b) Find the potential inside the box. $\rightarrow \nabla^2 V(x,y,z) = 0$

$x \rightarrow (x - a/2)$

$y \rightarrow (y - a/2)$





$z = 0 \rightarrow V_0$

$z = 0 \rightarrow 0$

$(V_{centre} \propto V_0)$ $V_c = \alpha . V_0$

$\boxed{V_c = \frac{V_0}{6}}$ $V_c = \frac{\alpha}{6} . V_0 < V_0$

Sol^n (a) All the 6 faces are located symmetrically w.r.t center.

(b) $$\frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0$$

$\rightarrow V(x,y,z) = f(x) \cdot g(y) \cdot h(z)$

$$\underbrace{\frac{1}{f(x)} \frac{d^2 f(x)}{dx^2}}_{= -k_1^2} + \underbrace{\frac{1}{g(y)} \frac{d^2 g(y)}{dy^2}}_{= -k_2^2} + \underbrace{\frac{1}{h(z)} \frac{d^2 h(z)}{dz^2}}_{k_3^2 = k_1^2 + k_2^2} = 0$$

$\Rightarrow f(x) = A \sin k_1 x + B \cos k_1 x$

$\Rightarrow g(y) = C \sin k_2 y + D \cos k_2 y$

$\Rightarrow h(z) = E e^{-k_3 z} + F e^{k_3 z}$

Boundary Cond^n $\rightarrow$

$V(x=0, y, z) = 0 \Rightarrow (B = 0)$

$V(x=a, y, z) = 0 \Rightarrow \boxed{k_1 = n\pi/a}$

$V(x, y=0, z) = 0 \Rightarrow (D = 0)$

$V(x, y=a, z) = 0 \Rightarrow \boxed{k_2 = m\pi/a}$

$V(x, y, z=0) = 0 \Rightarrow (E = -F)$

$V(x, y, z=a) = V_0$

$$V(x,y,z) = C_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{\sqrt{m^2+n^2}}{a} \pi z\right)$$

$$V(x,y,z) = C_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{\sqrt{m^2+n^2}\,\pi z}{a}\right)$$

$$V_0 = V(x,y,z=a) = C_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\sqrt{m^2+n^2}\,\pi\right)$$

$$\int_0^a \int_0^a V_0 \sin\left(\frac{n\pi x}{a}\right) \cdot \sin\left(\frac{m\pi y}{a}\right) \cdot dx \cdot dy = C_{mn} \sinh\left(\sqrt{m^2+n^2}\,\pi\right) \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \cdot dx \cdot \int_0^a \sin^2\left(\frac{m\pi y}{a}\right) \cdot dy$$

$$\frac{V_0 \left[1-(-1)^n\right]\left[1-(-1)^m\right]}{(n\pi/a)(m\pi/a)} = C_{mn} \cdot \frac{\sinh\left(\sqrt{m^2+n^2}\,\pi\right)}{4} \left[a^2\right].$$

$$C_{mn} = \frac{16 V_0}{\pi^2 mn} \cdot \frac{1}{\sinh\left(\sqrt{m^2+n^2}\,\pi\right)} \quad (m, n \text{ both odd})$$

$$= 0 \quad (\text{otherwise})$$

$$\boxed{V(x,y,z) = \sum_{n=1,3,5\ldots} \sum_{m=1,3,5\ldots} \frac{16 V_0}{\pi^2 mn} \cdot \frac{1}{\sinh\left(\sqrt{m^2+n^2}\,\pi\right)} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{\sqrt{m^2+n^2}\,\pi z}{a}\right)}$$

5. * Consider a spherical surface of a large radius R, the potential on the spherical surface is given below. Using the separation of variables find the potential $V(r,\theta)$ for $r \geq R$ upto order $O(1/r^6)$.

$$V(R,\theta) = \begin{cases} +V, & 0 \leq \theta < \frac{\pi}{2} \\ -V, & \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

$\cos\theta > 0 \qquad V > 0$

$\cos\theta < 0 \qquad V < 0$

$\longrightarrow$ Boundary Cond^n are odd (even) function of $\cos\theta$ then only odd (even) terms of legendre polynomials would survive

Sol^n $\nabla^2 V(r,\theta,\phi) = 0$

$\downarrow$

assume azimuthal symmetry $\Longrightarrow$ ϕ independent potential

$$V(r,\theta,\phi) = V(r,\theta) = f(r)\, g(\theta)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial (f(r)\cdot g(\theta))}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial (f(r)\cdot g(\theta))}{\partial\theta}\right) = 0$$

(Solutions)

$$\boxed{f(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}}$$

$g(\theta) = P_\ell(\cos\theta) \longrightarrow$ Legendre Polynomials

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$g_0(\theta) = 1$

$g_1(\theta) = \cos\theta$

$g_2(\theta) = \dfrac{3\cos^2\theta - 1}{2}$

$$\left\{ P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \right\} \rightarrow \text{Rodrigue's Formula}$$

First few Polynomials

$$\boxed{\begin{aligned} P_0(x) &= 1 \rightarrow \text{even} \\ P_1(x) &= x \rightarrow \text{odd} \\ P_2(x) &= \frac{3x^2-1}{2} \rightarrow \text{even} \\ P_3(x) &= \frac{5x^3-3x}{2} \rightarrow \text{odd} \end{aligned}}$$

$P_\ell(x)$ $\ell \to$ odd $P_\ell(x) \to$ odd

$\ell \to$ even $P_\ell(x) \to$ even

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)\left(P_\ell(\cos\theta)\right)}$$

$\ell = 5$

for $r \geq R$

$A_\ell = 0 \ \forall \ell \rightarrow$ otherwise pot. would blow up as $r \rightarrow \infty$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell \cdot P_\ell(\cos\theta)}{r^{\ell+1}}$$

Boundary Cond^n $\rightarrow$ $V(R,\theta) = V$ $\quad \theta \in (0, \pi/2)$

$V(R,\theta) = -V$ $\quad \theta \in (\pi/2, \pi)$

How to Find B_ℓ ??

Method -1 Use orthogonality of legendre polynomials

$$\int_{-1}^{1} P_\ell(x) \cdot P_m(x) \cdot dx = \frac{2}{2\ell+1}\delta_{\ell m}$$

$x = \cos\theta$

$$\Rightarrow \boxed{\int_0^{\pi} P_\ell(\cos\theta) \cdot P_m(\cos\theta) \cdot \sin\theta \, d\theta = \frac{2}{2\ell+1}\delta_{\ell m}}$$

$$V(R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} \cdot P_\ell(\cos\theta)$$

(m)

$\int_0^{\pi}$ ... $\cdot P_m(\cos\theta) \sin\theta \cdot d\theta$

$$\int_0^{\pi} V(R,\theta)\cdot P_m(\cos\theta)\cdot \sin\theta\ .d\theta = \sum_{l=0}^{\infty} \int_0^{\pi} \frac{B_l}{R^{l+1}} \underbrace{P_l(\cos\theta)\cdot P_m(\cos\theta)\ \sin\theta\ .d\theta}$$

$$\Rightarrow \int_0^{\pi/2} V\cdot P_m(\cos\theta)\ .\sin\theta\ .d\theta - \int_{\pi/2}^{\pi} V\cdot P_m(\cos\theta)\ \sin\theta\cdot d\theta = \frac{B_m}{R^{m+1}}\cdot \frac{2}{2m+1}$$

$$\int_0^{1} V\cdot P_m(x) - \int_{-1}^{0} V\, P_m(x)\cdot dx = \underbrace{\frac{B_m}{R^{m+1}}\cdot\frac{2}{2m+1}}$$

$x = -x$

$$\int_0^1 V\left(\underbrace{P_m(x) - P_m(-x)}\right)\cdot dx = \frac{B_m}{R^{m+1}}\cdot\frac{2}{2m+1}$$

$P_m(x) = P_m(-x)\quad \forall m \to \text{even}$

$\Rightarrow B_m = 0 \quad \forall m \to \text{even}$

$m = 1 \qquad P_m(x) = x \qquad \frac{3V\cdot R^2}{2} = B_1$

$m = 3 \qquad P_m(x) = \frac{5x^3 - 3x}{2} \qquad -\frac{7}{8}V\cdot R^4 = B_3$

$m = 5 \qquad P_m(x) = \frac{63x^5 - 70x^3 + 15x}{8} \qquad \frac{1}{4}\left[\frac{\cancel{63}^{21}}{\cancel{6}2} - \frac{70}{4} + \frac{15}{2}\right]\cdot\frac{11}{2}R^6 = B_5$

$\frac{11}{16}R^6 = B_5$

$$\Rightarrow V(r,\theta)\ (\text{upto order } 6) = \frac{3V}{2}\frac{R^2}{r^2}P_1(\cos\theta) - \frac{7}{8}V\frac{R^4}{r^4}P_3(\cos\theta) + \frac{11}{16}\frac{R^6}{r^6}P_5(\cos\theta)$$

Method-2 (For Solving for B_l)

↳ if boundary conditions are "nice" enough

eg. 6. * Let a sphere of radius R have potential $V(r = R, \theta, \phi) = V_0 \cos^2\theta$. Find the potential everywhere inside and outside the sphere.

$V(R,\theta) = V(R,\theta)$

$r = R$

$\boxed{B_l = 0\ \forall l} \qquad \Leftarrow V(r \le R, \theta) = \sum_{l=0}^{\infty} A_l r^l \cdot P_l(\cos\theta)$

$\Leftarrow V(r \ge R, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

$A_l R^l = \frac{B_l}{R^{l+1}} \quad \forall l$

$\boxed{\frac{B_l}{A_l} = R^{2l+1}} \quad -(1)$

$$\Leftarrow \; V(r \geq R, \theta) = \sum_{l=0} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$\boxed{A_l}$

Boundary Cond^n $\longrightarrow$

$$V(r=R, \theta) = V_0 \cos^2\theta = A_0 + A_1 \cdot R(\cos\theta) + A_2 R^2 \frac{(3\cos^2\theta - 1)}{2}$$

↳ others are zero ($A_3, A_4 --$)

$$\Rightarrow \frac{3A_2R^2}{2} = V_0$$

$$\Rightarrow \boxed{A_2 = \frac{2V_0}{3R^2}} \qquad \boxed{A_1 = 0}$$

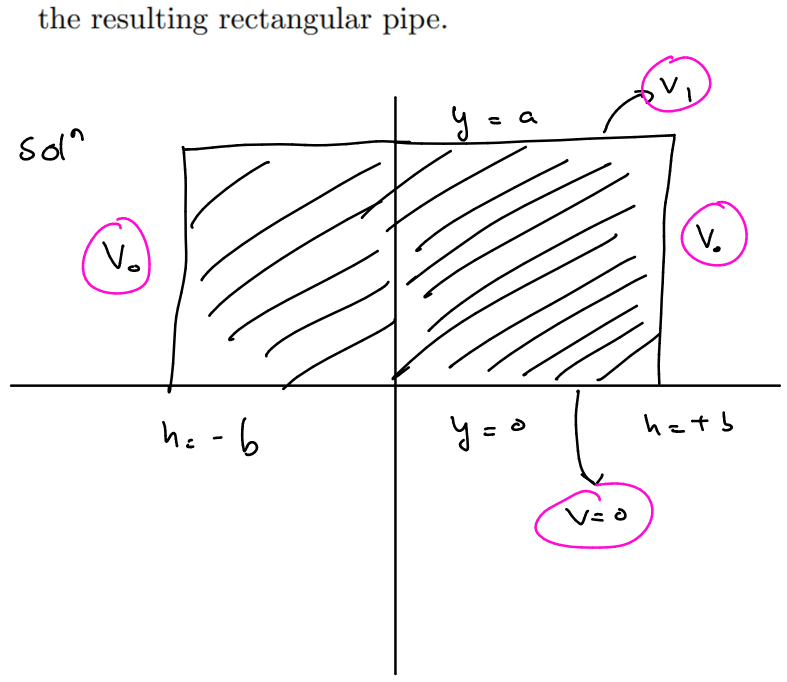
$$A_0 = \frac{A_2 R^2}{2} \Rightarrow \boxed{A_0 = \frac{V_0}{3}}$$

$$\Rightarrow \boxed{V_{inside} = \frac{V_0}{3} + \frac{V_0 r^2}{3R^2}(3\cos^2\theta - 1)}$$

$$\Rightarrow \boxed{V_{outside} = \frac{V_0}{3}\frac{R}{r} + \frac{V_0}{3}\frac{R^3}{r^3}(3\cos^2\theta - 1)}$$

4. * Two infinitely long metal plates at y = 0 and y = a are connected at x = ±b by metal strips maintained at a constant potential V_0. The potential on the bottom (y = 0) is zero, however the potential on the top (y =a) is a nonzero constant V_1. A thin layer of insulation at each corner prevents the plates from shorting out. Find the potential inside the resulting rectangular pipe.

Sol^n



$$V(x,y) = f(x)\cdot g(y)$$

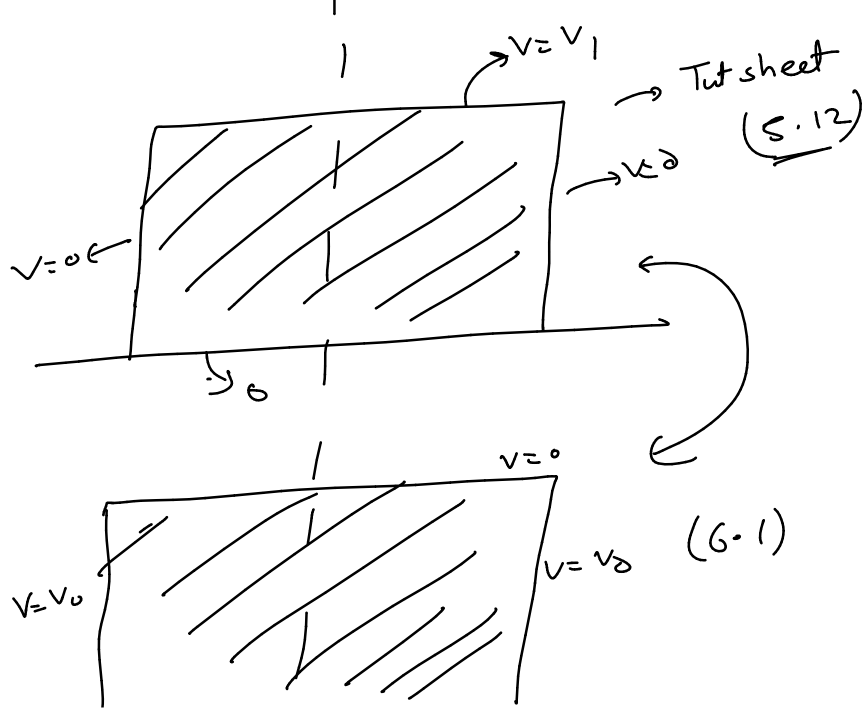
$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} = 0$$

$$\qquad \| \qquad\qquad\qquad \|$$

$$\quad -k^2 \qquad\qquad\quad k^2$$

$$f(x) = A\sin kx + B\cos kx$$

$$g(y) = Ce^{ky} + De^{-ky}$$



Boundary Cond^n $\longrightarrow$ (i) $V(x, y=0) = 0 \Rightarrow C = -D$

(ii) $V(x=b, y) = V(x=-b, y) = V_0$

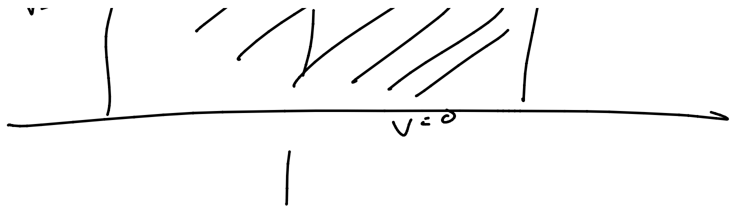
$\Downarrow$

$A = 0$

$\neq \; (B\cos kb)(\sinh ky) = V_0 \;\; \forall y$

↳ $\neq 0$

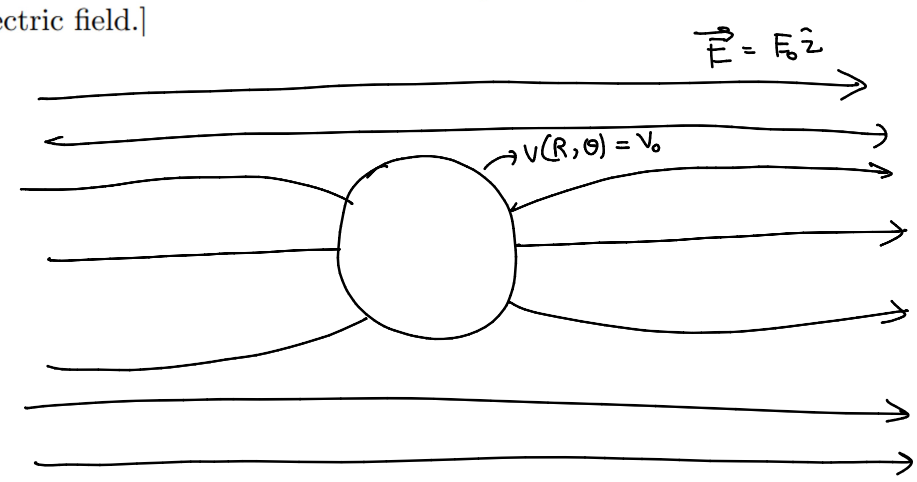
Potential



Potential is not variable separable

7. An uncharged, conducting sphere of radius R is placed in a region where the electric field is uniform i.e. $\vec{E} = \vec{E}_0$. Can you guess l value will be odd or even in potential outside the sphere? Find the electric field in the region after the sphere is put in place. [Hint:(a)Use spherical polar co-ordinate solution of Laplace equation. (b)Ponder over what boundary condition will be used to find coefficient a_l and b_l and take derivative to find electric field.]

Sol^n



At the surface of sphere

$V(R,\theta) =$ Constant $= V_0$ (say)

(even func^n of $\cos\theta$)

$\downarrow$

l-value will be even

Boundary Cond^{n}s (i) $V(r \le R, \theta) = V_0$

(ii) $V(r \to \infty, \theta) = -E_0 r \cos\theta$