Tutorial 7 Solution

17 May 2022 12:04

1. Consider a thin spherical shell (thickness $\to 0$) of radius R with a surface charge density;

$$\sigma(\theta) = \sigma_0(cos\theta + cos^2\theta)$$

2. Using solutions of Laplace's equation, find the potential $V(r, \theta)$ everywhere, both for r > R and r < R.

Sol^n Using std. solutions for Laplace's Eqn

$$V_1(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r<R)$$

$$V_2(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (r>R)$$

Boundary Conditions

① $V_1(R,\theta) = V_2(R,\theta)$

$$\Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \quad \forall l$$

$$\Rightarrow \boxed{B_l = A_l R^{2l+1}}$$

② $$\varepsilon_0 \left.\frac{\partial V_1(r,\theta)}{\partial r}\right|_{r=R} - \varepsilon_0 \left.\frac{\partial V_2(r,\theta)}{\partial r}\right|_{r=R} = \sigma$$

$$\varepsilon_0 \left[\sum_{l=0}^{\infty} l A_l r^{l-1} P_l(\cos\theta) + \sum_{l=0}^{\infty} \frac{(l+1) B_l}{r^{l+2}} P_l(\cos\theta) \right] = \sigma_0(\cos\theta + \cos^2\theta)$$

$A_l, B_l = 0 \quad \forall l>2$

$$\varepsilon_0 \left[\frac{B_0}{R^2} + \left(A_1 + \frac{2B_1}{R^3}\right)\cdot\cos\theta + \left(2A_2 R + \frac{3B_2}{R^4}\right)\left(\frac{3\cos^2\theta - 1}{2}\right) \right] = \sigma_0(\cos\theta + \cos^2\theta)$$

Using $B_l = A_l R^{2l+1}$

$B_0 = A_0 R, \quad B_1 = A_1 R^3, \quad B_2 = A_2 R^5$

$$\varepsilon_0 \left[\frac{A_0}{R} - \frac{1}{2}(5A_2 R) + (3A_1)\cos\theta + \frac{(15A_2 R)\cos^2\theta}{2} \right] = \sigma_0(\cos\theta + \cos^2\theta)$$

$$\Rightarrow \boxed{A_1 = \frac{\sigma_0}{3\varepsilon_0}} \quad \boxed{A_2 = \frac{2\sigma_0}{15\varepsilon_0 R}} \quad \boxed{A_0 = \frac{\sigma R}{3\varepsilon_0}}$$

$$V_1(r,\theta) = \frac{\sigma R}{3\varepsilon_0} + \frac{\sigma_0 r}{3\varepsilon_0}\cos\theta + \frac{2\sigma_0 r^2}{15\varepsilon_0 R}\left(\frac{3\cos^2\theta - 1}{2}\right) \qquad (r<R)$$

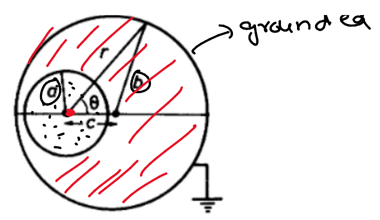
$$V_2(r,\theta) = \frac{\sigma_0 R^2}{3\varepsilon_0 r} + \frac{\sigma_0 R^3}{3\varepsilon_0 r^2}\cos\theta + \frac{2\sigma_0 R^4}{15\varepsilon_0 r^3}\left(\frac{3\cos^2\theta - 1}{2}\right)$$

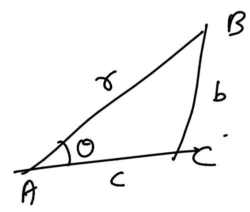
ea

2. *In the following system (see figure), the inner conducting sphere of radius a carries charge Q and the outer sphere of radius b is grounded. The distance between the centres is c which is a small quantity.

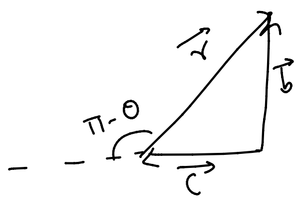
   (a) Show that to the first order in c, the equation describing the outer sphere, using the centre of inner sphere as origin, is r(θ)=b+c.cosθ.

   (b) If the potential between two spheres contains only $l = 0$ and $l = 1$ angular components, determine it to first order in c.





Sol^n (a)



$$\vec{c} + \vec{r} = \vec{b}$$

$$r^2 - 2rc\cos\theta + c^2 = b^2$$

$$r^2 - 2rc\cos\theta + (c^2 - b^2) = 0$$

$$r = \frac{2c\cos\theta \pm \sqrt{4b^2 - 4c^2\sin^2\theta}}{2}$$

$$r = c\cos\theta \pm b\sqrt{1 - \frac{c^2\sin^2\theta}{b^2}} \qquad b \gg c$$

$$r = c\cos\theta \pm b\left(1 - \cancel{\frac{c^2\sin\theta}{2b^2}}\right)$$

$$= \boxed{r = c\cos\theta + b}$$

($\because b > c$ for $r > 0$, $+$ sign is considered)

(b)

$$V(r,\theta) = \sum_{\ell=0,1}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

r ↓ distance from origin of

$A_0,\ B_0,\ A_1,\ B_1$

Boundary Conditions → ① $V(r = b + c\cos\theta, \theta) = 0 \quad \forall\,\theta$ Outer sphere being grounded

$$\left(A_0 + \frac{B_0}{b + c\cos\theta}\right) + \left(A_1\cdot(b + c\cos\theta) + \frac{B_1}{(b + c\cos\theta)^2}\right)\cos\theta = 0 \quad \forall\,\theta$$

$$\left[A_0 + \frac{B_0}{b}\left(1 - \frac{c\cos\theta}{b}\right)\right] + \left[A_1(b + c\cos\theta) + \frac{B_1}{b^2}\left(1 - \frac{2c\cos\theta}{b}\right)\right]\cos\theta \quad \forall\,\theta$$

$$\left(A_0 + \frac{B_0}{b}\right) + \left(A_1 b + \frac{B_1}{b^2} + \frac{B_0}{b}\right)\cos\theta + O(\cos^2\theta) + O(c^2) = 0 \quad \forall\,\theta$$

$$\Rightarrow \boxed{A_0 + \frac{B_0}{b} = 0} \quad -(1) \qquad \boxed{A_1 b + \frac{B_1}{b^2} + \frac{B_0}{b} = 0} \quad -(2)$$

②
$$\boxed{-\epsilon_0 \oint \frac{\partial V}{\partial r}\bigg|_{r=a}(dA) = Q}$$

Whole inner sphere

$$\epsilon_0\left(\frac{\partial V}{\partial r}\bigg|_{inside} - \frac{\partial V}{\partial r}\bigg|_{outside}\right) = \sigma_0$$

$$E_{outside} - E_{inside} = \frac{\sigma_0}{\epsilon_0}$$

$$E_{outside} = \frac{\sigma_0}{\epsilon_0}$$

③ $V(r = a, \theta) = \text{constant}$

$$\boxed{\frac{\partial V}{\partial \theta}\bigg|_{r=a,\theta} = 0}$$

$$\left.\frac{\partial V}{\partial \theta}\right|_{r=a,\theta} = -$$

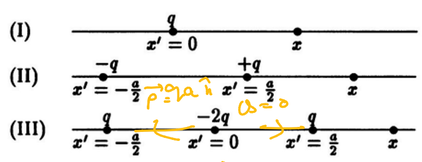
3. *Static charges are distributed along the x-axis (one-dimensional) in the interval $-a \le x' \le a$. The charge density is :

$dQ = \rho(x')\,dx'$ line charge density

$$\rho(x') \quad for \quad |x'| \le a \qquad (-a, a)$$
$$0 \quad for \quad |x'| > a$$

- Write down the multipole expansion for the electrostatic potential $\phi(x)$ at a point x on the axis in terms of $\rho(x')$, valid for $x > a$.
- For each charge configuration given in Fig. 2, find (a) total charge $Q = \int \rho dx'$, (b) dipole moment $P = \int x' \rho dx'$.

$Q = q \quad \vec{P} = 0$



$Q = 0, \quad \vec{P} = 0$

Figure 2: Charge distributions for Q.4

$\rho(x') = 0 \quad \forall \quad x < -a, \; x > a$

$\left(\because \; x > a \;\&\; x' \in (-a, a) \right)$
$\quad \frac{x'}{x} \in (-1, 1)$

Sol^n

$$\phi(x) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho(x')\cdot dx'}{|x - x'|} = \frac{1}{4\pi\epsilon_0} \int_{-a}^{a} \frac{\rho(x')\cdot dx'}{x\left|1 - \frac{x'}{x}\right|}$$

loc^n of point where potential is calculated

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{x} \int_{-a}^{a} \frac{\rho(x')}{1 - \frac{x'}{x}}\cdot dx'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{x}\left[\int_{-a}^{a} \rho(x') \left[1 + \left(\frac{x'}{x}\right) + \left(\frac{x'}{x}\right)^2 + \cdots \infty \right] \cdot dx' \right]$$

$$\Rightarrow \boxed{\phi(x) = \frac{1}{4\pi\epsilon_0} \sum_{n=0} \frac{1}{x^{n+1}} \int_{-a}^{a} \rho(x') \cdot (x')^n \cdot dx'}$$

$|\vec{r}|^{n+1}$

$\rho(\vec{r}')\cdot d^3\vec{r}' \quad (|\vec{r}'|^n)$

(I) $\displaystyle Q = \int_{-\infty}^{\infty} \rho(x')\, dx' = \int_{-\infty}^{\infty} q\,\delta(x')\cdot dx' = q$

$\displaystyle P = \int_{-\infty}^{\infty} \rho(x')\cdot x' \cdot dx' = \int_{-\infty}^{\infty} q\,\delta(x')\cdot x'\cdot dx' = 0$

(II) $\displaystyle Q = \int_{-\infty}^{\infty} \rho(x')\cdot dx' = \int_{-\infty}^{\infty} -q\,\delta(x'+a/2) + q\,\delta(x'-a/2)\, dx' = -q + q = 0$

$\displaystyle P = \int_{-\infty}^{\infty} \rho(x')\cdot x' \cdot dx' = \int_{-\infty}^{\infty} -q\,\delta(x+a/2)\cdot x' + q\,\delta(x-a/2)\cdot x' \cdot dx' = -q\left(-\frac{a}{2}\right) + q\left(\frac{a}{2}\right)$

$= qa$

(III) $Q = 0$

$P = 0$

4. A circular disc of radius R lies in the $z = 0$ plane, centred at the origin. It has the following charge density frozen on it;

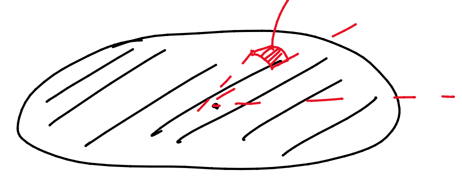
$$\sigma(r', \phi) = \sigma_0 r' \cos(\phi)$$

(a) What is the monopole moment of the configuration?

(b) Calculate the dipole contribution to the potential due to the configuration at $(0, 0, z)$ using the expression in polar form.

(c) Now calculate the the cartesian components of the dipole moment of the configuration. Use this to calculate the dipole contribution at $(0, 0, z)$. Verify your answer with the expression obtained in (b)

Sol^n (a)



$$dQ = \sigma \cdot dA = \sigma\,(dr')(r'\,d\phi) = \sigma_0 (r')^2 \cos\phi \, d\phi \cdot dr'$$

$$Q = \int_0^{2\pi}\int_0^{R} \sigma_0 (r')^2 \cdot dr' \cdot \cos\phi \, d\phi$$

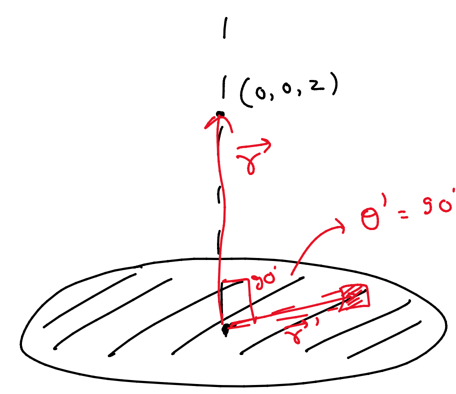
$$\boxed{Q = \frac{\sigma_0 R^3}{3} \times 0 = 0}$$

(b) $$V_{dipole} = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^2} \int \rho(r') \cdot d^3 r' \cdot r' \cdot \underbrace{\cos\theta'}_{\to 0} = 0$$

$\theta' \to \angle$ b/w $\vec{r}'$ & $\vec{r}$

$\vec{r}'$ ↳ location of source charge

$\vec{r}$ ↳ location of the point where potential is calculated.



$\theta' = 90°$ ∀ all elements

(c) $$\vec{p} = \int (\vec{r}')\,\underbrace{(\rho(\vec{r}'))\,(d^3\vec{r}')}_{dQ' = \sigma(\vec{r}')(d^2\vec{r}')}$$

$$= \int_0^{2\pi}\int_0^{R} (r'\,\hat{r}')\left[\sigma_0 r' \cos\phi \cdot (r') \cdot (dr')(d\phi)\right]$$

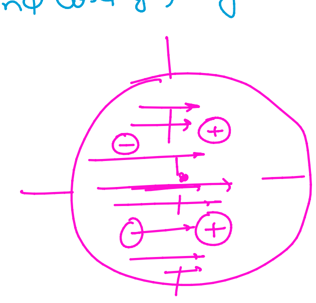
↳ not a constant unit vector

$$= \cancel{0}$$

$$= \int_0^{2\pi}\int_0^{R} (r'\cos\phi\,\hat{x} + r'\sin\phi\,\hat{y})\left[\sigma_0 (r')^2 \cos\phi \, dr' . d\phi\right]$$

$$= \frac{\sigma_0 R^4}{4}\int_0^{2\pi} (\cos^2\phi\,\hat{x} + \sin\phi\cos\phi\,\hat{y})\,d\phi$$

$$\boxed{\vec{p} = \frac{\sigma_0 R^4}{4} \cdot \pi\hat{x}}$$



Net dipole moment $\hat{x}$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{\sigma_0 \pi R^4}{4}\hat{x}\right)\cdot(z\,\hat{z})}{z^2} = 0$$

6. * Find the dipole moment of:

(a) A ring with charge per unit length $\lambda = \lambda_0 \cos\phi$ where ϕ is the angular variable in cylindrical coordinates.

(b) a sphere with charge per unit areas $\sigma = \sigma_0 \cos\theta$ where θ is the polar angle measured from the positive z-axis.

2. * Problem 6 (a) and (b): Calculate the potential for both the cases up to the quadrupole term.

$$\rho(\vec{r}\,') = \lambda\, \delta(r' - R) \cdot \delta(z')$$

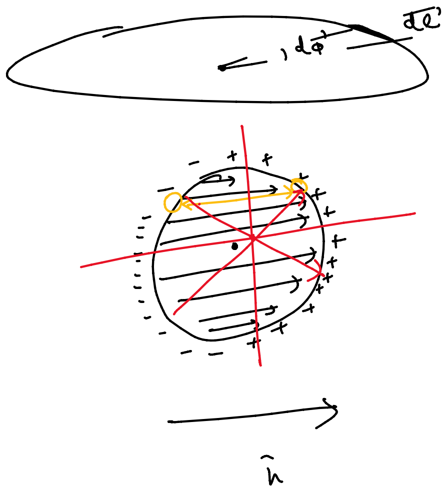
Sol (a) $\vec{p} = \int \rho(\vec{r}\,') \cdot d^3 r'\ \vec{r}\,' = \int \boxed{\lambda(\vec{r}\,')(R\, d\phi')\ \vec{r}\,'} \Big/ \rho(\vec{r}\,')\, dr' \cdot r' d\phi' \cdot dz$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} \lambda\, \delta(r' - R) \cdot r'\, dr' d\phi' dz'\ \delta(z') = \int_0^{2\pi} (\lambda R \cdot d\phi)$$

scalar $\leftarrow$ $M = Q = \int \rho(\vec{r}\,') \cdot d^3 r'$

$dl' = R\, d\phi$

$$= \int_0^{2\pi} \lambda_0 \cos\phi' \cdot R \cdot d\phi' \left[R \cos\phi' \hat{n} + R \sin\phi' \hat{y} \right]$$



$$\boxed{\vec{p} = \lambda_0 R^2 \cdot \pi\ \hat{n}}$$

$\vec{r} = (r, \theta, \phi)$

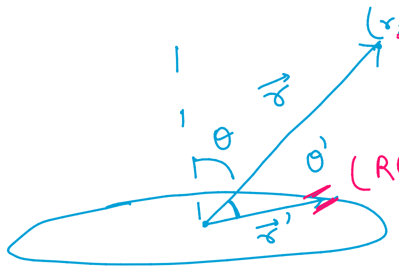
$\dfrac{KQ}{|\vec{r}|} = \dfrac{KQ}{r}$

Total charge on ring $Q = 0 \rightarrow$ monopole term $= 0$

$$V(\vec{r}) = \frac{K \vec{p} \cdot \hat{r} \cdot |\vec{r}|}{|\vec{r}|^2 |\vec{r}|} + K \frac{1}{|\vec{r}|^3} \int \lambda(\vec{r}\,', \phi) \cdot (d\vec{r}\,') |\vec{r}\,'|^2 \left(\frac{3\cos^2\theta' - 1}{2} \right)$$

$$V(\vec{r}) = \frac{K(\lambda_0 R^2 \pi\, \hat{n}) \cdot (r \sin\theta \cos\phi\, \hat{n} + r \sin\theta \sin\phi\, \hat{y} + r\cos\theta\, \hat{z})}{|\vec{r}|^3}$$

$(\cos^2\theta')$ $\angle$ b/w $\vec{r}\,'$ & $\vec{r}$



(r, θ, ϕ)

$(R \cos\phi' \hat{n} + R \sin\phi' \hat{y})$

$(R, \frac{\pi}{2}, \phi')$

$$+ \frac{K}{|\vec{r}|^3} \int \underbrace{(\lambda_0 \cos\phi')(R\, d\phi')}_{Q} \frac{R^2}{|\vec{r}\,'|^2} \left(\frac{3\cos^2\theta' - 1}{2} \right)$$

$\dfrac{Q R^2}{2}$

$$\cos\theta' = \frac{(r \sin\theta \cos\phi\, \hat{n} + r \sin\theta \sin\phi\, \hat{y} + r \cos\theta\, \hat{z}) \cdot (R \cos\phi' \hat{n} + r' \sin\phi' \hat{y})}{r \cdot R}$$

$$\cos\theta' = (\sin\theta \cos\phi \cos\phi' + \sin\theta \sin\phi \sin\phi') = \sin\theta \cos(\phi - \phi')$$

$$\text{Quadrupole term} = \frac{K}{r^3} \int (\lambda_0 \cos\phi')(R^3 d\phi') \left[\frac{3}{2} (\sin^2\theta \cos^2\phi \cos^2\phi' + \sin^2\theta \sin^2\phi \sin^2\phi' + 2\sin^2\theta \cos\phi \sin\phi \cos\phi' \sin\phi') - \cancel{\frac{1}{2}} \right]$$

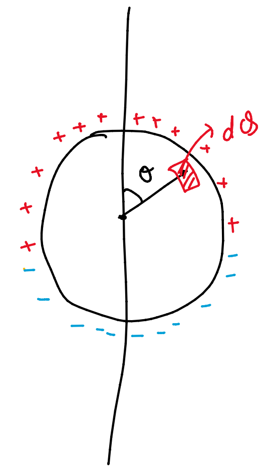
$$= \frac{K}{r^3} \int_0^{2\pi} \frac{3}{2} R^3 \lambda_0 \left[\sin^2\theta \cos^2\phi\ \cancel{\cos^3\phi'}^{\,0} + \sin^2\theta \sin^2\phi\ \cancel{\sin^2\phi' \cos\phi'} + 2\sin^2\theta \cos\phi \sin\phi\ \cancel{\cos^2\phi' \sin\phi'} \right] \cdot d\phi'$$

$$= 0$$

$$\boxed{V(\vec{r}) = K \lambda_0 \pi \left(\frac{R}{r} \right)^2 \sin\theta \cos\phi + O(\bar{r}^4)}$$

$\llcorner$ special case if $\theta = 0$ (point on z-axis) $\rightarrow V(\vec{r}) = 0 + O(\bar{r}^4)$

(b)



$d\mathcal{Q} = \sigma_\circ R^2 \sin\theta\, d\theta\, d\phi$

$\rho(\vec{r}\,')(r')^2 \sin\theta'\, dr'\, d\theta'\, d\phi$

$$\vec{P} = \int \rho(\vec{r}\,').\, d^3\vec{r}\,'.\, (\vec{r}\,') \qquad \iiint \sigma(\vec{r}\,')\,\delta(r'-R)\,(r')^2 \sin\theta'\, dr'\, d\theta\, d\phi'$$

$r'\sin$

$$\iint \sigma(R)\, R^2 \sin\theta'\, d\theta'\, d\phi'$$

$$= \int \sigma(R).\, d^2\vec{r}\,'\; (\vec{r}\,')$$

$$\vec{P} = \int (\sigma_0 \cos\theta')\,(R^2 \sin\theta'\, d\theta'\, d\phi') \quad (R\sin\theta' \cos\phi'\,\hat{n} + R\sin\theta' \sin\phi'\hat{y} + R\cos\theta'\,\hat{z})$$

$\vec{r}$

$$\vec{P} = \sigma_0 R^3 \int_0^{2\pi}\int_0^{\pi} \cancel{(\sin^2\theta' \cos\theta' \cos\phi')\hat{n}.\, d\theta'\, d\phi'} + \cancel{(\sin^2\theta' \cos\theta' \sin\phi)\hat{y}\, d\theta'\, d\phi'} + (\cos^2\theta' \sin\theta')\hat{z}.\, d\theta'.\, d\phi'$$

$$\vec{P} = \sigma_0 R^3 . 2\pi \quad 2\int_0^{\pi/2} \sin\theta' \cos^2\theta'.\, d\theta' \quad \hat{z}$$

$$\vec{p} = 4\pi\sigma_0 R^3 \frac{1}{3\cdot 1} = \boxed{\frac{4\pi\sigma_0 R^3}{3}\hat{z}}$$

Monopole term $\rightarrow Q_{total} = 0$ (due to symmetric charge dist. in the upper & lower hemisphere)

Dipole term $\rightarrow$ $$K\frac{\vec{P}.\vec{r}}{|\vec{r}|^3} = K\frac{\left(\frac{4\pi\sigma_0 R^3}{3}\right)\cdot(r\cos\theta)}{r^3} = \frac{4\pi K\sigma_0}{3}\left(\frac{R^3}{r^2}\right)\cos\theta$$

r^2

Quadrupole term

$$\frac{K}{|\vec{r}|^3}\int \rho(\vec{r}\,')\, d^3\vec{r}\,'\; |\vec{r}\,'|^2 \left(\frac{3\cos^2\varphi' - 1}{2}\right)$$

$$\frac{K}{r^3}\int (\sigma_0\cos\theta')\,(R^2 \sin\theta'\, d\theta'\, d\phi') \cdot R^2 \left(\frac{3\cos^2\varphi' - 1}{2}\right)$$

$\cos\varphi' = \angle b/w\ \vec{r}\ \&\ \vec{r}\,' = \hat{r}\cdot\hat{r}\,'$

$= \sin\theta \sin\theta' \cos\phi\cos\phi' + \sin\theta . \sin\theta' \sin\phi \sin\phi' + \cos\theta \cos\theta'$

$= \sin\theta \sin\theta' \cos(\phi - \phi') + \cos\theta \cos\theta'$

$2\pi \quad \pi \quad [3\cos^2\varphi - 1]$

$$\frac{K}{r^3}(\sigma_0 R^4)\int_0^{2\pi}\int_0^{\pi}\sin\theta'\cos\theta'\,d\theta'\,d\phi'\left[\frac{3\cos^2\varphi}{2}-\cancel{\frac{1}{2}}\right]$$

$$\frac{K\sigma_0 R^4}{r^3}\cdot\frac{3}{2}\int_0^{2\pi}\int_0^{\pi}(\sin\theta'\cos\theta')\Big[\sin^2\theta\sin^2\theta'\cos^2(\phi-\phi')+\cos^2\theta\cos^2\theta' + 2\sin\theta\sin\theta'\cos\theta\cos\theta'\cos(\phi-\phi')\Big]\,d\theta'\,d\phi'$$

$$\frac{3}{2}K\sigma_0 R\left(\frac{R}{r}\right)^3\int_0^{2\pi}\int_0^{\pi}\cancel{\sin^3\theta'\cos\theta'\sin^2\theta\cos^2(\phi-\phi')} + \cancel{\cos^2\theta\cos^3\theta'\sin\theta'} + \cancel{2\sin\theta\cos\theta\sin^2\theta'\cos^2\theta'\cos(\phi-\phi')}\,d\theta'\,d\phi'$$

$$= 0$$

$$\boxed{V(r,\theta,\phi) = \frac{4}{3}\pi K\sigma_0 R\left(\frac{R}{r}\right)^2\cos\theta + o(r^{-4})}$$

↳ These terms also vanish

7. * An electric dipole of moment $P = (P_x, 0, 0)$ is located at the point $(x_0, y_0, 0)$ where $x_0 > 0$ and $y_0 > 0$. The planes $x = 0$ and $y = 0$ are conducting plates with a tiny gap at the origin. The potential of the plate at $x = 0$ is maintained at V_0 and the plate at $y = 0$ is grounded. The dipole is sufficiently weak so that you can ignore the charges induced on the plates.

(a) Based on Fig 3, deduce a simple expression for the electrostatic potential $\phi(x, y)$.

(b) Calculate the force on the dipole.

$\theta \in (0, \pi/2)$

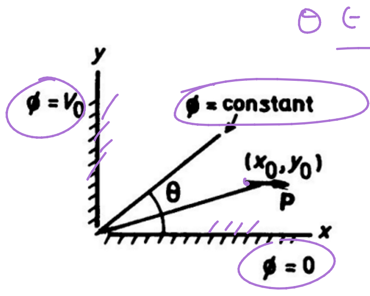
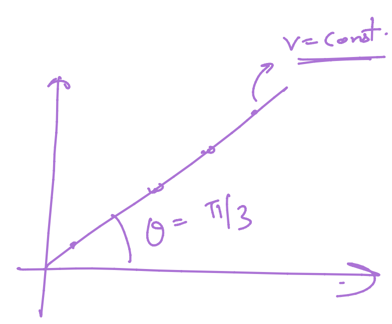


Figure 3: Plates at an angle θ.

$x_0-\delta x,\ x_0+\delta x$



Sol^n $\boxed{\nabla^2 V(r,\theta) = 0}$ (since δ is negligible)

↳ From the figure $V(r,\theta) = V(\theta)$

$$\frac{1}{r^2}\frac{\partial^2 V(\theta)}{\partial\theta^2} = 0$$

$$\boxed{V(\theta) = A\theta + B}$$

(a) $V(0) = 0$

$V(\pi/2) = V_0$

$$\boxed{V(\theta) = \frac{2V_0\theta}{\pi}}$$

(b) $\vec{U} = -\vec{p}\cdot\vec{E}$

$\vec{F} = -\nabla\cdot\vec{U}$

(b) U

$$\vec{F} = -\vec{\nabla}\cdot\vec{U}$$

$$= -\vec{\nabla}\cdot(-\vec{p}\cdot\vec{E})$$

$$= \vec{\nabla}\cdot(\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\cdot\vec{E}$$

$$\underline{\underline{\vec{\nabla}\cdot(\vec{A}\cdot\vec{B})}}$$