Tutorial 9 Solution

08 June 2022 11:51

2. * What current density $\vec{J}$ would produce the vector potential, $\vec{A} = k\hat{\phi}$ (where k is a constant), in cylindrical coordinates?

given $\vec{A}$ find $\vec{J}$

Sol^n

$\nabla \cdot \vec{B} = 0 \rightarrow$ No magnetic monopoles $\rightarrow$ Can define $\vec{A}$ such that $\boxed{\vec{B} = \nabla \times \vec{A}}$ $\rightarrow$ mag vector potential

(analogue of electric potential Φ)

$\nabla \times \vec{B} = \mu_0 \vec{J}$

$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$

Recall from Helmholtz theorem to determine a vector we need it's curl & divergence both.

given $\nabla \times \vec{A} = \vec{B}$

$\nabla \cdot \vec{A} \stackrel{?}{=} \longrightarrow$ freedom to choose $\nabla \cdot \vec{A} \rightarrow$ gauge freedom

Lorentzian gauge

Most common Gauge $\boxed{\nabla \cdot \vec{A} = 0}$ (Columb gauge)

$\Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$ $\qquad (\nabla^2 \Phi = -\rho/\epsilon_0)$

In cartesian coords can be decomposed into

$\boxed{\nabla^2 A_{x,y,z} = -\mu_0 J_{x,y,z}} \rightarrow 3 \text{ eq}^n$

$\hookrightarrow \nabla^2 A_\phi = -\mu_0 J_\phi$??

Reason $\nabla^2(\vec{V}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(V_x \hat{x} + V_y \hat{y} + V_z \hat{z})$

Laplacian of a vector

$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(V_r \hat{r} + V_\theta \hat{\theta} + V_z \hat{z})$

Variable unit vectors

$\rightarrow$ Even if $\vec{A}$ was given in cartesian coordinates using $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ is wrong. Why ??

$$\rightarrow \vec{B} = \nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{r} h_1 & \hat{\theta} h_2 & \hat{z} h_3 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 h_1 & A_2 h_2 & 0 h_3 \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \hat{r} & \hat{\theta} r & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & kr & 0 \end{vmatrix}$$

$$\begin{bmatrix} & & \\ 0\,h_1 & k\,h_2 & 0\,h_3 \end{bmatrix} \qquad \begin{bmatrix} & & \\ 0 & kr & 0 \end{bmatrix}$$

$$= \frac{1}{r}\left(k\,\hat{z}\right) = \frac{k}{r}\hat{z}$$

$$\rightarrow \quad \mu_0 \vec{J} = \nabla \times \vec{B} = \frac{1}{r}\begin{bmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{k}{r} \end{bmatrix} = \frac{1}{r}\left(r\,\hat{\theta}\,\frac{k}{r^2}\right) = \frac{k}{r^2}\hat{\theta}$$

$$\boxed{\vec{J} = \frac{k}{\mu_0 r^2}\,\hat{\theta}} \qquad \vec{A} = k\,\hat{\theta} \qquad (\theta \equiv \phi)$$

3. * A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$ where k is a constant and $\vec{r}$ is the vector from the center.
   (a) Calculate the bound charges σ_b and ρ_b
   (b) Find the electric field inside and outside the sphere.

Sol^n $\vec{P} \rightarrow$ dipole moment per unit volume $\rightarrow \frac{d\vec{p}}{d\tau}$

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{d\vec{p}\,' \cdot \hat{\mathscr{r}}}{|\mathscr{r}|^2} \qquad \mathscr{r} = |r - r'| \rightarrow \frac{Kr\cos\theta}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{|\mathscr{r}|^2}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \vec{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{|\mathscr{r}|}\right)d\tau' \qquad \left(\nabla\cdot(f\vec{A}) = \vec{A}\cdot(\nabla f) + f(\nabla\cdot\vec{A})\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \nabla'\cdot\left(\vec{P}(\vec{r}\,')\frac{1}{|\mathscr{r}|}\right)\cdot d\tau' - \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\mathscr{r}|}\nabla'\cdot\vec{P}(\vec{r}\,')\cdot d\tau$$

$$= \frac{1}{4\pi\epsilon_0}\oint \frac{\vec{P}(\vec{r}\,')\cdot d\vec{a}\,'}{|\mathscr{r}|} - \frac{1}{4\pi\epsilon_0}\int \frac{1}{|\mathscr{r}|}\nabla'\cdot\vec{P}(\vec{r}\,')\,d\tau$$

Define $\vec{P}(\vec{r})\cdot\hat{n} = \sigma_b(\vec{r})$ $\quad -\nabla\cdot\vec{P}(\vec{r}) = \rho_b$

↳ $\vec{P}$ at surface

Define $\rho(\vec r) \cdot (\hat n) = \sigma_b$ ...

$\hookrightarrow \vec P$ at surface

$$= \frac{1}{4\pi\epsilon_0} \underbrace{\int_A \frac{\sigma_b(\vec r')}{|r - r'|} da}_{} + \frac{1}{4\pi\epsilon_0} \underbrace{\int_V \frac{\rho_b(\vec r')\, d\tau}{|r - r'|}}_{}$$

(a) $\hat n = \hat r$

$$\sigma_b = \vec P \cdot \hat n = (k\vec r)\cdot(\hat r) = (kR\hat r)\cdot(\hat r) = kR$$

On surface

$$\rho_b = -\nabla \cdot \vec P = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, kr) = -\frac{3kr^2}{r^2} = -3k$$

$$\vec P(\vec r) = k\vec r$$

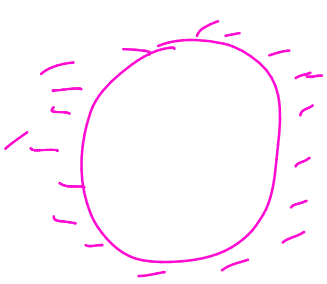
(b)

$V = V_\sigma + V_\rho$

$$E_{inside} = E_\sigma + E_\rho \longrightarrow \frac{\rho r}{3\epsilon_0} = \frac{-3kr\,\hat r}{3\epsilon_0} = \frac{-kr}{\epsilon_0}\hat r$$

$E_\sigma \downarrow 0$

(Uniform surface charge density)



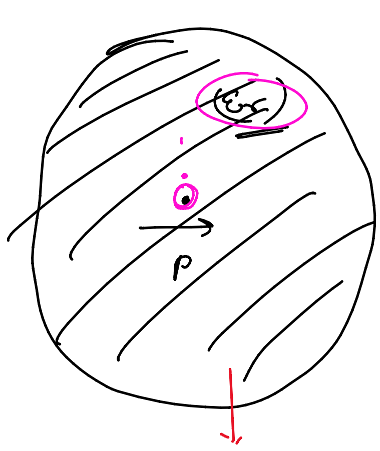
$$E_{outside} = E_\sigma + E_\rho = \frac{1}{4\pi\epsilon_0}\frac{(kR \cdot 4\pi R^2)}{r^2} + \frac{1}{4\pi\epsilon_0}\frac{\left(-3k \times \frac{4}{3}\pi R^3\right)}{r^2} = 0$$

Total charge induced $= 0$

$\boxed{\vec D = \epsilon_0 \epsilon_r \vec E} \rightarrow$ linear dielectric

6. * Find the electric potential inside and outside a homogeneous linear isotropic dielectric sphere (with dielectric constant ϵ_r and radius R), at the centre of which a pure dipole $\vec p$ is imbedded.

Sol^n



Apply Laplace's eq^n in a region excluding the centre.

Use std. sol's to Laplace's Eq^n

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty}\left(\underbrace{A_l^{(in)} r^l}_{} + \frac{B_l^{(in)}}{r^{l+1}}\right)(P_l(\cos\theta))$$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty}\left(A_l^{(out)} r^l + \frac{B_l^{(out)}}{r^{l+1}}\right) P_l(\cos\theta)$$

Boundary Cond^n

(i) $\boxed{\lim_{r\to\infty} V_{out}(r,\theta) = 0} \Rightarrow \boxed{A_l^{(out)} = 0 \quad \forall l}$

(ii) $\lim_{r\to 0} V_{in}(r,\theta) = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{p\cos\theta}{r^2} \Rightarrow B_l^{(in)} = 0 \;\forall l \neq 1$

$B_1^{(in)} = \frac{p}{4\pi\epsilon_0\epsilon_r}$

(σ_f)

(iii) $V_{in}(R,\theta) = V_{out}(R,\theta)$

$E_{\cdot} \quad E_{\cdot} = \sigma_f$

(iii) $V_{in}(R,\theta) = V_{out}(R,\theta)$

(iv) $D_{\perp out} - D_{\perp in} = \sigma_f = 0$

$\boxed{E_{\perp out} - E_{\perp in} = \frac{\sigma_b}{\epsilon_0}}$

(for conductors)

$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$

$\boxed{\vec{D} = \epsilon_0 \epsilon_r \vec{E}}$

$\epsilon_0 E_{\perp out} - \epsilon_0 \epsilon_r E_{\perp in} = 0$

$$-\epsilon_0 \frac{\partial V_{out}}{\partial r}\bigg|_{r=R} + \epsilon_0 \epsilon_r \frac{\partial V_{in}}{\partial r}\bigg|_{r=R} = 0$$

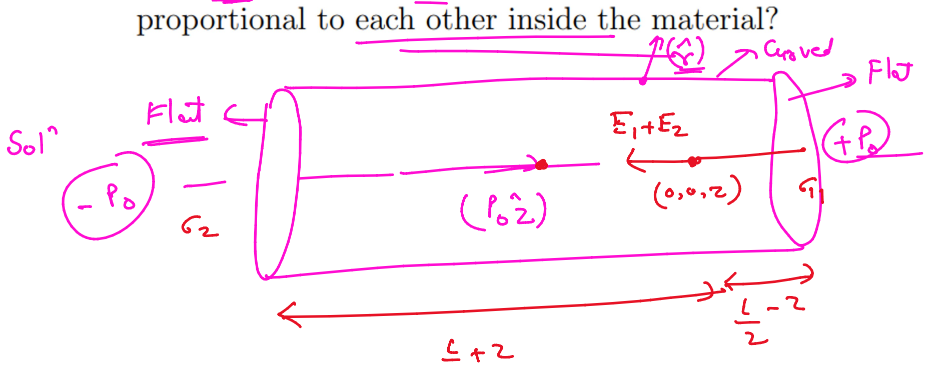
$$\sum_{l=0}^{\infty} \frac{B_l^{(out)}}{r^{l+1}} \cdot P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l^{(in)} r^l \cdot P_l(\cos\theta) + \frac{p\cos\theta}{4\pi\epsilon_0\epsilon_r r^2}$$

(iv) $$+\cancel{\epsilon_0} \sum_{l=0}^{\infty} \frac{-B_l^{(out)}(l+1) \cdot P_l(\cos\theta)}{r^{l+2}} = \cancel{\epsilon_0}\epsilon_r \sum_{l=0}^{\infty} l A_l^{(in)} r^{l-1} P_l(\cos\theta) - \frac{2p\cos\theta}{4\pi\epsilon_0\epsilon_r r^3}$$

(7)

7. * A cylinder of radius R and height L is positioned such that the origin is at the center and the z-axis is along the axis of the cylinder. The cylinder carries a frozen polarisation $\vec{P} = P_o\hat{z}$. Calculate $\vec{E}$ and $\vec{D}$ at all points on the z-axis. Are the quantities $\vec{D}$ and $\vec{E}$ proportional to each other inside the material?

Sol^n



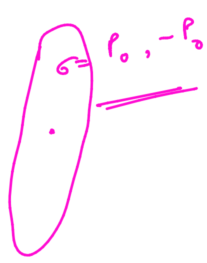
$(\sigma_b)_{Total} = P_0\hat{z} \times (\hat{z}) + P_0\hat{z}(-\hat{z}) = 0$

$\rho_b = -\nabla \cdot \vec{P} = 0$

There are individual charge densities on the two flat faces

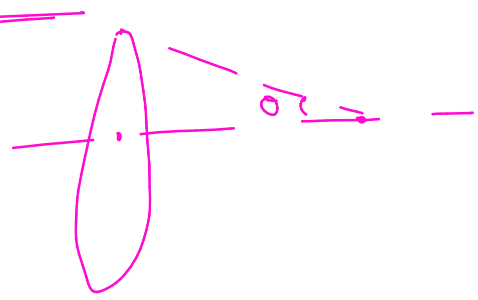
$\sigma = \pm P_0$

$\Rightarrow \vec{P} \cdot \hat{n} = \hat{z} \cdot \hat{r} = 0$



Once we obtain σ the system is just a combination of two discs with given uniform surface charge densities.

$$\left(\vec{E}_{disc}\right)_{on\ axis} = \frac{\sigma}{2\epsilon_0}(1 - \cos\theta_0)$$



$\vec{E}_{inside} = \vec{E}_1 + \vec{E}_2$

$$= \left[\frac{P_0}{2\epsilon_0}\left(1 - \frac{(L/2 - z)}{\sqrt{\left(\frac{L}{2}-z\right)^2 + R^2}}\right) + \frac{P_0}{2\epsilon_0}\left(1 - \frac{\left(\frac{L}{2}+z\right)}{\sqrt{\left(\frac{L}{2}+z\right)^2 + R^2}}\right)\right] \cdot -\hat{z}$$

$$E_{outside}(z > L/2) = \left[\frac{P_0}{2\epsilon_0}\left(\frac{z - L/2}{\sqrt{}} - \frac{z + L/2}{\sqrt{}}\right)\right]\hat{z}$$

$E_{inside}(z < -L/2) = -E_{outside}(z > L/2)$

$\vec{D}_{inside} = \epsilon_0(\vec{E}_{inside}) + P_0$

$\vec{D}_{outside} = \epsilon_0 \vec{E}_{outside} \rightarrow \parallel$ only in this case

$D_{outside} = \varepsilon_0 E_{outside}$

8. * A conducting sphere of radius R is half submerged in a linear, homogeneous, semi-infinite liquid dielectric medium of doelectric constant κ. The sphere is at a potential V_0. Assuming there is no bound charge at the liquid-air interface, calculate

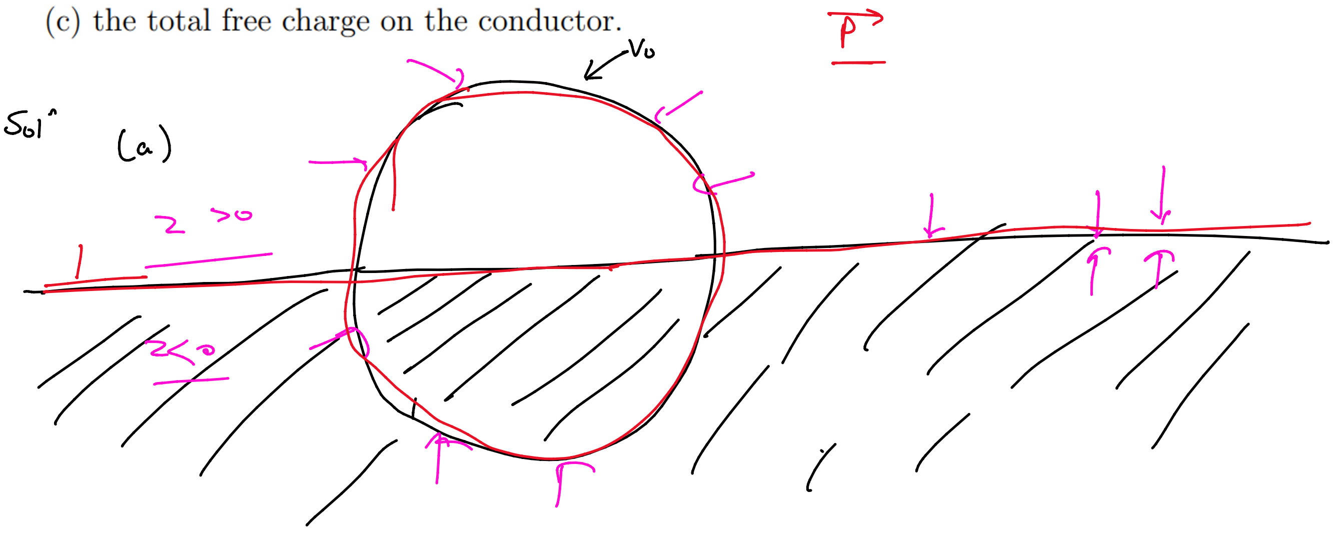
(a) the potential at a point outside the sphere,

(b) the electric field, the electric dispacement, and the surface and volume bound charge density in the dielectric,

(c) the total free charge on the conductor.

$\vec{P}$

Sol^n (a)



(a) We can again use sol^ns to Laplace's Eq^n

$r>R$

$$V^{+}(r, \theta < \pi/2) = \sum_{l=0}^{\infty}\left(A_l^+ r^l + \frac{B_l^+}{r^{l+1}}\right) P_l(\cos\theta)$$

$$V^{out}(r, \theta > \pi/2) = \sum_{l=0}^{\infty}\left(A_l^- r^l + \frac{B_l^-}{r^{l}}\right) P_l(\cos\theta)$$

Boundary Conditions

(i) → $A_l^+ = A_l^- = 0 \quad \forall l$ $\quad \left(\lim_{r\to\infty} V(r) = 0\right)$

(ii) $V(\theta = \pi/2^+) = V(\theta = \pi/2^-)$

$\theta = \pi/2 \quad P_l(\cos\theta) = 0 \quad \forall l \to odd$ $\quad P_l(0)$

→ $\boxed{B_l^+ = B_l^-}$ $(\forall l \to even)$

(iii) $V^+(r=R, \theta) = V^-(r=R, \theta) = V_0$

$\Rightarrow B_l^+ = B_l^- \quad \forall l$

& $B_l^+ = B_l^- = 0 \quad \forall l \neq 0$

$\Rightarrow \boxed{B_l^+ = B_l^- = V_0 R}$ $\quad (l=0)$

$$\boxed{V(r,\theta) = \frac{V_0 R}{r}}$$

(b) $E(r,\theta) = -\nabla \cdot V = +\frac{V_0 R}{r^2}\hat{r}$

$(\quad) \quad (+V_0R)\hat{r} = \varepsilon_0 \vec{E} + \vec{P}$

(b) r^2

$$D(r,\theta)_{z<0} = \varepsilon_0\varepsilon_r\left(+\frac{V_0R}{r^2}\right)\hat{r} = \varepsilon_0\vec{E} + \vec{P}$$

$$D(r,\theta)_{z>0} = \varepsilon_0\left(\frac{V_0R}{r^2}\right)\hat{r}$$

outside dielectric

$$\vec{P} = 0 \quad (z>0)$$

$$\vec{P} = \varepsilon_0(\varepsilon_r-1)\left(\frac{V_0R}{r^2}\right)\hat{r} \quad (z<0)$$

$$\sigma_b = \vec{P}\cdot\hat{n} \longrightarrow \text{Two surfaces}$$

$r = R$

$$\sigma_b(r=R,\theta)_{z<0} = \varepsilon_0(\varepsilon_r-1)\frac{V_0}{R}$$

$$\sigma_b(r=R,\theta)_{z>0} = 0$$

$$\sigma_b(z=0) = (\ \ )\hat{r}\cdot\hat{z} = 0$$

$$\rho_b(r>R) = -\nabla\cdot\vec{P} = -\varepsilon_0(\varepsilon_r-1)V_0R\left(\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)\right) = \boxed{0} \quad (\text{for } \forall z)$$

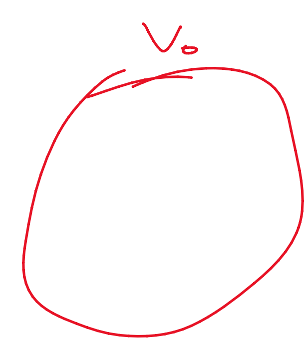
$\hookrightarrow 4\pi\delta(r)$

(c) Free charge on sphere

$$\boxed{\nabla\cdot\vec{D} = \rho_f} \rightarrow \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_b+\rho_f}{\varepsilon_0}$$

$$\int \vec{D}\cdot ds = Q_f$$

$$Q_f = \varepsilon_0\varepsilon_r\frac{V_0R}{R^2}\times 2\pi R^2 + \varepsilon_0\frac{V_0R}{R^2}\times 2\pi R^2$$



$$\boxed{Q_f = 2\pi\varepsilon_0V_0R(1+\varepsilon_r)}$$

$\varepsilon_r = 1$ $\quad 4\pi\varepsilon_0V_0R$