Gravitational Waves in Precessing Rigid Bodies

Course Project: Gravitational Waves Physics and Astronomy

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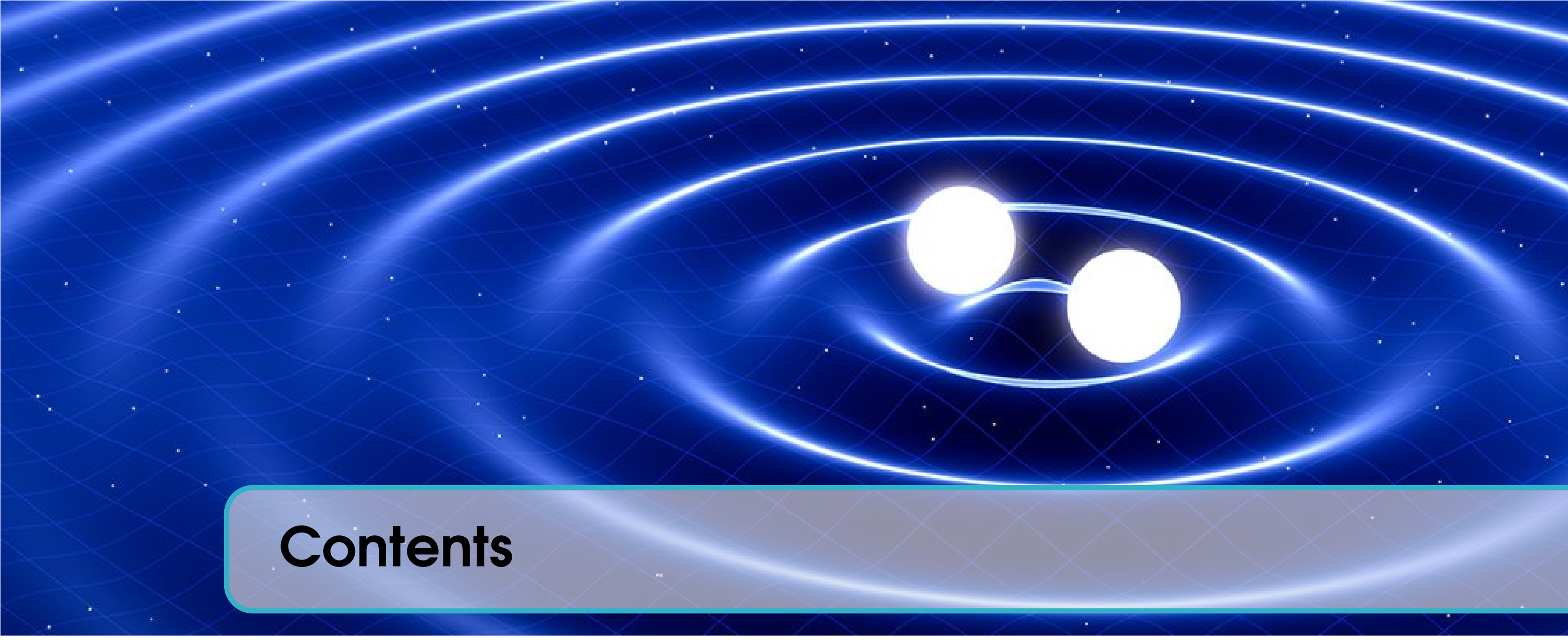
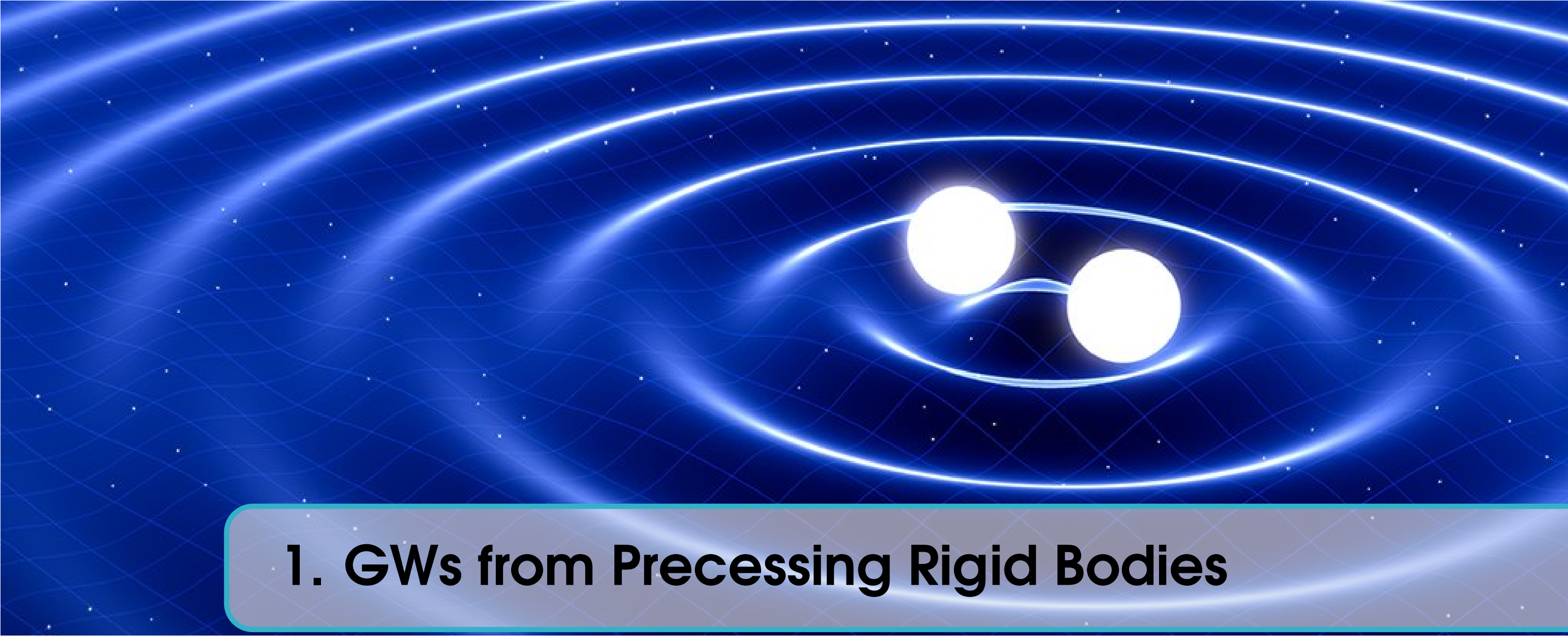


Contents



1. GWs from Precessing Rigid Bodies

1.1 Introduction

In this report I've analysed the emission of Gravitational Waves (GWs) from precessing rigid bodies. This analysis is an extension of a similar analysis done in class where Prof Archana Pai performed the calculation for a rigid body rotating about its own axis.
I've first summarised the results of the calculations done in the class and then introduced the notion of Euler Angles which is very helpful in studying such systems. Then I computed the moment of inertia tensor in the fixed reference frame which is used to calculate the GW amplitude and Power Radiated in GW by the system. At the end I calculate the value of amplitude for pulsar, which is a very common astrophysical precessing rigid body. I conclude this report by summarising my results.

1.2 GWs from rotation around a principal axis: Summary

For a rigid body rotating about its own axis the GW amplitude is given by

$$h_+ = h_o \frac{1+\cos^2 i}{2} \cos(2\pi f_{GW} t) \tag{1.1}$$
$$h_\times = h_o \sin(2\pi f_{GW} t) \tag{1.2}$$
$$ \tag{1.3}$$

where

$$h_o = \frac{4\pi^2 G}{c^4} \frac{I_3 f_{GW}^2}{r} \varepsilon \tag{1.4}$$
$$h_o \sim 1.06 \times 10^{-24} \left(\frac{f_{gw}}{1Khz}\right)^2 \left(\frac{kpc}{r}\right) \tag{1.5}$$
$$\left(\frac{I_3}{10^{38} kgm^2}\right) \left(\frac{\varepsilon}{10^{-6}}\right) \tag{1.6}$$

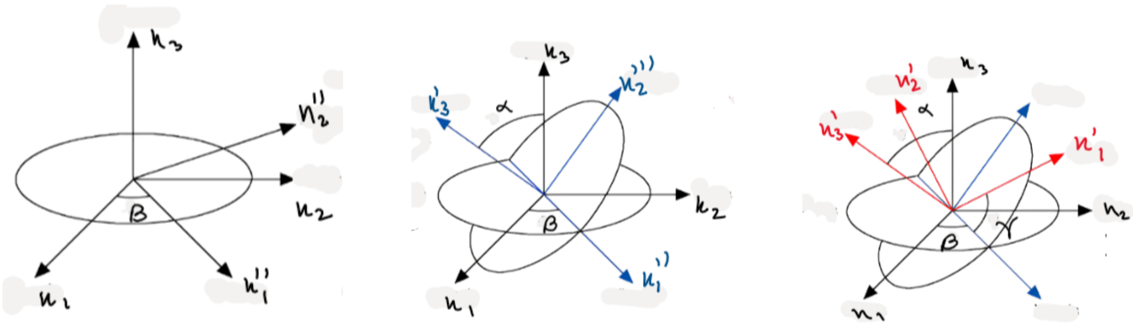
In the above equations $\varepsilon = \frac{I_1 - I_2}{I_3}$ is ellipticity and I_1, I_2, I_3 are moments of inertia about the principal axes of the body. In the body frame, which is a refernce frame attached to the body, the moment of inertia tensor is diag(I_1, I_2, I_3).

The power radiated through these GWs is given by

$$
\begin{aligned}
L_{GW} &= \frac{2}{5}\frac{G}{c^5}\langle \dddot{I}_{11}^2 + \dddot{I}_{12}^2 \rangle \\
L_{GW} &= \frac{32G}{5c^5}\varepsilon^2 I_3^2 \omega_{rot}^6 \qquad (1.7)
\end{aligned}
$$

1.3 Euler Angles and their time evolution

The notion of Euler angles is especially helpful in the study of those systems which are not rotating about their principal axes. We first consider a fixed reference frame (x_1, x_2, x_3). The rigid body rotates about the axis x_3 and hence angular momentum $\mathbf{J}$ is conserved in this reference frame and points along x_3 direction. Next we introduce the body frame, i.e. a reference frame attached to the rotating body, with coordinates (x_1', x_2', x_3') whose axes coincide with the principal axes of the body. The relation between the two frames is given by the Euler angles (α, β, γ) as shown below



(a) Rotation about x_3 (b) Rotation about x_1'' (c) Rotation about x_3'

Figure 1.1: Euler Angles

The tranformation between Body Frame and Fixed Frame is given by

$$x_i' = \mathscr{R}_{ij} x_j \qquad (1.8)$$

where $\mathscr{R}$ is the Rotation matix given by

$$\mathscr{R} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1.9)$$

$$
\begin{aligned}
\mathbf{J} &= (0, 0, J) && \text{(Fixed Frame)} \qquad (1.10) \\
&= (J_1', J_2', J_3') && \text{(Body Frame)} \qquad (1.11)
\end{aligned}
$$

From 1.1 we get

$$
\begin{aligned}
J_1' &= J \sin\alpha \sin\gamma \qquad (1.12) \\
J_2' &= J \sin\alpha \cos\gamma \qquad (1.13) \\
J_3' &= J \cos\alpha \qquad (1.14)
\end{aligned}
$$

and from the definition of principal axis

$$
\begin{aligned}
J_1' &= I_1 \omega_1' \qquad (1.15) \\
J_2' &= I_2 \omega_2' \qquad (1.16) \\
J_3' &= I_3 \omega_3' \qquad (1.17)
\end{aligned}
$$

Here $\omega' = (\omega_1', \omega_2', \omega_3')$ is the total angular velocity in primed coordinates. ω' is also equal to the sum of rate of change of angles α, β, γ

$$\omega' = (\omega_1', \omega_2', \omega_3') \tag{1.18}$$

$$= \frac{d\alpha}{dt} + \frac{d\beta}{dt} + \frac{d\gamma}{dt} \tag{1.19}$$

$$= (\dot{\alpha}\cos\gamma, -\dot{\alpha}\sin\gamma, 0) \tag{1.20}$$

$$+ (\dot{\beta}\sin\alpha\sin\gamma, \dot{\beta}\sin\alpha\cos\gamma, \dot{\beta}\cos\alpha)$$

$$+ (0, 0, \dot{\gamma})$$

Combinig (1.2-1.17) and (1.20) we get

$$I_1(\dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma) = J\sin\alpha\sin\gamma \tag{1.21}$$

$$I_2(-\dot{\alpha}\sin\gamma + \dot{\beta}\sin\alpha\cos\gamma) = J\sin\alpha\cos\gamma \tag{1.22}$$

$$I_3(\dot{\beta}\cos\alpha + \dot{\gamma}) = J\cos\alpha \tag{1.23}$$

We can obtain $\alpha(t), \beta(t)\text{and}\gamma(t)$ by integrating the above equations. In the most general case, the result can be written in terms of elliptic functions. However, in this report we'll limit our analysis to the simpler case of an axisymmetric body with $I_1 = I_2$

1.3.1 "Wobble" radiation from axisymmetric rigid body

We consider an axisymmetric body, whose longitudinal axis x_3' makes an angle α with the angular momentum axis x_3. The angle α is often called the "wobble" angle and the corresponding GW emission is called the "wobble radiation".

For $I_1 = I_2$ the analytic solutions of eqn $1.21 - 1.23$ are quite simple.

- Multiplying 1.21 by $\cos\gamma$ and 1.22 by $\sin\gamma$ and adding both gives

$$I_1\dot{\alpha} = 0 \tag{1.24}$$

$$\dot{\alpha} = 0 \tag{1.25}$$

- Substituting 1.25 in 1.21 we get

This shows that, in the body frame, the angular velocity rotates in the (x_1', x_2') i.e. precesses around the x_3', with angular velocity ω_p. Observe that $|I_3 - I_1| << I_3$ (in a Neutron Star, a possible value could be $|I_3 - I_1|/I_3 \sim 10^{-7}$), and therefore $|\omega_p| << \Omega$. This motion is called free precession, since it takes place in the absence of external torques, just as a consequence of the deviation of the rigid body from spherical symmetry. To visualise this precession more clearly check this link out

1.4 GW amplitude for precessing rigid bodies

The gravitational wave perturbation tensor 'h' is given by

$$h \sim \frac{2G}{rc^4}\ddot{I} \tag{1.26}$$

So we first calculate how the moment of inertia tensor I evolves with time

1.4.1 Inertia Tensor Calculation

In the body frame moment of inertia tensor is $I_{ij}' = diag(I_1, I_2, I_3)$ and is constant wrt time. Using I_{ij}' and the rotation matrix $\mathscr{R}(= A(\gamma)B(\alpha)C(\beta))$ as defined in 1.3 we can calulate the tensor I_{ij} in the fixed frame.

$$
\begin{aligned}
I_{ij} &= (\mathscr{R}^T I' \mathscr{R})_{ij} \\
I &= (C^T B^T A^T) I' (ABC) \\
I &= (C^T B^T) I' (BC)
\end{aligned}
\tag{1.27}
$$

where the last equation follows from the fact that A commutes with I' since $I_1 = I_2$ and since A is orthogonal $A^T A = 1$ Now we evaluate the remaining matrix computation

$$
\begin{aligned}
I &= \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \\
&\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1\cos^2\alpha + I_3\sin^2\alpha & (I_1 - I_3)\cos\alpha\sin\alpha \\ 0 & (I_1 - I_3)\cos\alpha\sin\alpha & I_1\sin^2\alpha + I_3\cos^2\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} I_1\cos\beta & -(I_1\cos^2\alpha + I_3\sin^2\alpha)\sin\beta & -(I_1 - I_3)\cos\alpha\sin\alpha\sin\beta \\ I_1\sin\beta & (I_1\cos^2\alpha + I_3\sin^2\alpha)\cos\beta & (I_1 - I_3)\cos\alpha\sin\alpha\cos\beta \\ 0 & (I_1 - I_3)\cos\alpha\sin\alpha & I_1\sin^2\alpha + I_3\cos^2\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{1.28}
$$

$$
I_{11} = I_1(\cos^2\beta + \cos^2\alpha\sin^2\beta) + I_3\sin^2\sin^2\beta = \frac{1}{2}(I_1 - I_3)\sin^2\alpha\cos 2\beta + constant
\tag{1.29}
$$

$$
I_{12} = I_{21} = \frac{1}{2}(I_1 - I_3)\sin^2\alpha\sin 2\beta \tag{1.30}
$$

$$
I_{13} = I_{31} - (I_1 - I_3)\cos\alpha\sin\alpha\sin\beta \tag{1.31}
$$

$$
I_{22} = I_1(\sin^2\beta + \cos^2\alpha\cos^2\beta) + I_3\sin^2\cos^2\beta = \frac{1}{2}(I_1 - I_3)\sin^2\alpha\cos 2\beta + constant
\tag{1.32}
$$

$$
I_{23} = I_{32} = (I_1 - I_3)\cos\alpha\sin\alpha\cos\beta \tag{1.33}
$$

$$
I_{33} = I_1\sin^2\alpha + I_3\cos^2\alpha = constant \tag{1.34}
$$

The only time dependence in the above equations arises due to $\beta = \Omega t$ as all the other quantities α, I_1, I_2, I_3 are constants. Double differentiating above inertia tensor components w.r.t time we obtain

$$
\ddot{I}_{11} = 2(I_3 - I_1)\Omega^2\sin^2\alpha\cos(2\Omega t) \tag{1.35}
$$

$$
\ddot{I}_{12} = 2(I_3 - I_1)\Omega^2\sin^2\alpha\sin(2\Omega t) \tag{1.36}
$$

$$
\ddot{I}_{22} = 2(I_1 - I_3)\Omega^2\sin^2\alpha\cos(2\Omega t) \tag{1.37}
$$

$$
\ddot{I}_{13} = (I_1 - I_3)\Omega^2\sin\alpha\cos\alpha\sin(\Omega t) \tag{1.38}
$$

$$
\ddot{I}_{23} = -(I_1 - I_3)\Omega^2\sin\alpha\cos\alpha\cos(\Omega t) \tag{1.39}
$$

$$
\ddot{I}_{33} = 0 \tag{1.40}
$$

1.4.2 Amplitude Calculation

For a GW radiation in a direction corresponding to polar angles $\theta = i, \phi = 0$ the inertia tensor is given by rotating the coordinate system (x_1, x_2, x_3) using the rotation matrix

$$\mathscr{R}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix}$$
$$\ddot{I}'' = \mathscr{R}^T \ddot{I} \mathscr{R}$$
$$Wedirectlyusetheresultofthiscomputationdoneinclass \tag{1.41}$$

Using 1.26 and comparing with

$$\begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(Assuming a z polarized wave)} \tag{1.42}$$

we obtain

$$h_+ = \frac{G}{rc^4}(\ddot{I}_{11} - \ddot{I}_{22}\cos^2 i + \ddot{I}_{23}\sin 2i - \ddot{I}_{33}\sin^2 i)$$
$$= A_{+,1}\sin(\Omega t) + A_{+,2}\sin(2\Omega t) \tag{1.43}$$
$$h_\times = \frac{G}{rc^4}(\ddot{I}_{12} - \ddot{I}_{13}\sin i)$$
$$= A_{\times,1}\sin(\Omega t) + A_{\times,2}\sin(2\Omega t) \tag{1.44}$$

where

$$A_{+,1} = h_0' \sin 2\alpha \sin i \cos i \tag{1.45}$$
$$A_{+,2} = 2h_0' \sin^2\alpha (1 + \cos^2 i) \tag{1.46}$$
$$A_{\times,1} = h_0' \sin 2\alpha \sin i \tag{1.47}$$
$$A_{\times,2} = 4h_0' \sin^2\alpha \cos i \tag{1.48}$$
$$\tag{1.49}$$

and

$$h_0' = \frac{G}{c^4}\frac{I_3 - I_1\Omega^2}{r} \tag{1.50}$$

Eqns $1.43 - 1.44$ give us the expression for GW amplitude for "+" and "×" polarisations respectively
Some remarkable features of these GWs are

- Unlike the binary merger case or the bodies rotating about their principal axis, the GWs from precessing rigid bodies consist of two frequency components $\omega_g w = \Omega$ and $\omega_g w = 2\Omega$
- Observe that substituting wobble angle $\alpha = 0$ in $1.43 - 1.44$ results in no GW emission. This is also expected since $\alpha = 0$ is the case when the body rotates about its principal axis and for such an axisymmetric body there is no GW emission

1.5 Power radiated through GWs

We can calculate the power radiated in these GWs using the relation

$$L_{GW} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I}_{11}^2 + \dddot{I}_{22}^2 + 2\dddot{I}_{12}^2 + 2\dddot{I}_{23}^2 + 2\dddot{I}_{13}^2 \rangle$$

$$L_{GW} = \frac{2G}{5c^5}(I_1 - I_3)^2\Omega^6(\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha) \tag{1.51}$$

The first term in L_{GW} is from the radiation at $\omega_g w = \Omega$ while the second term is from the radiation at $\omega_g w = 2\Omega$. The two contributions become equal when $\cos^2\alpha = 16\sin^2\alpha$ i.e.at $\alpha \approx 14^\circ$. For smaller vaues of α, the contribution from $\omega_{gw} = \Omega$ wave is higher than the $\omega_{gw} = 2\Omega$ wave

1.6 GWs from Pulsar

Pulsars are the most commonly known astrophysical sources know to mankind that precess around their axis. Pulsars are known to emit radiations in radio and x-ray regime. And their rotation axis is different from their emission axis which leads to the detction of a periodic signal from them. As we just saw in the above analysis that a precessing rigid body emits GWs, we'll try to estimate the strength of these GWs from pulsars and compare them with the GWs emitted from a typical Neutron Star rotating about its axis. From 1.50 we have

$$h_o \sim \frac{G}{rc^4}\varepsilon I_3\Omega^2 \tag{1.52}$$

where ε is ellipticity defined as

$$\varepsilon = \frac{I_3 - I_1}{I_3}$$

$$h_o \sim \frac{4\pi^2 G}{rc^4}\frac{\varepsilon I_3}{P^2} \tag{1.53}$$

$$h_o \sim 1.06 \times 10^{-24}\left(\frac{ms}{P}\right)^2\left(\frac{kpc}{r}\right)\left(\frac{I_3}{10^{38}kgm^2}\right)\left(\frac{\varepsilon}{10^{-6}}\right) \tag{1.54}$$

As computed in $1.1c$ the order of amplitude is same for a typical Neutron Star rotating about its axis and having similar mass. We calculate this amplitude for three pulsars -:

- For the Crab Pulsar P = 33ms and r = 2kpc

$$h_o^{Crab} = 4.7 \times 10^{-28}\left(\frac{I_3}{10^{38}kgm^2}\right)\left(\frac{\varepsilon}{10^{-6}}\right)$$

- For the Vela Pulsar P = 89ms and r = 0.5kpc

$$h_o^{Vela} = 2.6 \times 10^{-28}\left(\frac{I_3}{10^{38}kgm^2}\right)\left(\frac{\varepsilon}{10^{-6}}\right)$$

- For the millisecond pulsar PSR 1957+20 Pulsar P = 1.61ms and r = 1.5kpc

$$h_o^{1957+20} = 2.7 \times 10^{-25}\left(\frac{I_3}{10^{38}kgm^2}\right)\left(\frac{\varepsilon}{10^{-6}}\right)$$

For millisecond pulsars $\varepsilon \sim 10^{-9}$ hence the amplitude of GWs from these pulsars is also of the order$\sim 10^{-28}$.

The strength of the signals is very low as compared to the typical sensitivity of current detectors (10^{-21}) hence GWs from these sources can't be detected using our current GW detectors.

1.7 Summary

In this report we saw how even axisymmetric rigid bodies not rotating about their principal axis can produce GW. Theses GWs from freely precessing rigid bodies consist of two frequency components one equal to the frequency of rotation and the other equal to twice the frequency of rotation of the rigid body. The power in the low-frequency component dominates at lower wobble angle(α) values and as the wobble angle is increased the power in the higher-frequency component starts dominating. The amplitude and power radiated through these GWs are very less for their detection by current detectors and these quantities are respectively of the same order as GWs produced from bodies rotating about their principal axis.

1.8 References

- "Gravitational Waves Volume 1 Theory and Experiments"- Michele Maggiore
- "Gravitational waves from pulsars: emission by the magnetic field induced distortion" Bonazzola et al 1996