Summer of Science 2021

Maths n Physics Club

IIT Bombay

Elementary Particle Physics

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July 20, 2021

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1 Introduction to Elementary Particles

1.1 Origins

The birth of elementary particle physics was the discovery of electron in 1897 by J.J. Thomson.Then came Rutherford's α particle scattering experiment which established the presence of a dense positive charge sitting right at the centre of an atom.In 1914 Niels Bohr proposed a model for hydrogen consisting of a single electron revolving around the proton.Bohr conducted experiments to verify his hypothesis and the results remarkably agreed with the predictions for a hydrogen atom.But it was soon found out that the nucleus of heavier atoms like Helium,Lithium weighed more than they would have if they consisted only of protons .The dilemma was resolved in 1932 by Chadwick's discovery of neutron-an electrically neutral twin to the proton.

By 1932,though only 3 elementary particles(e^-, p^+, n) were discovered but ideas for existence of others were already being discussed and three major ones were -Yukawa's meson,Dirac's positron and Pauli's neutrino

1.2 Mesons

The discovery of mesons is credited to the very fundamental question "What prevents a nucleus from getting torn apart?".To resolve the issue Physicists came up with the prediction of 'Strong Force'(a short range force between nucleons which supresses the effect of electrostatic repulsion between protons).Yukawa in 1934 came up with the first significant theory of strong forces and proposed that neutron and proton are attaracted to each other by some sort of a *field* just similar to previously known electrostatic or gravitational fields.He argued that this field should also be quantized and the corresponding quantum was named mesons(which as we'll see further turned out to be wrong) ,precisely because it was predicted to be around 300 times heavy as compared to an e^- and 1/6 th the mass of a proton which makes its mass lying between the lighter particles(leptons) and heavier particles(baryons).

Experiments conducted on cosmic rays to study and detect these Yukawa's mesons ultimately led to the discovery of two middle weight particles-the lighter and longer lived "muon"(μ) and the heavier and shorter lived "pion"(π). Further studies revealed that muon behaved as a heavier version of electron in many ways and thus classified as lepton.Pions were the true Yukawa mesons which takes part in strong interactions.

1.3 Antiparticle

The first major achievement in the field of 'Relativistic Quantum Mechanics' was discovery of Dirac's equation in 1927 by Paul Dirac.The equation was supposed to describe free electrons with energy given by the relativistic formula $E^2 - p^2c^2 = m^2c^4$ .But it had a problem: For every positive-energy solution ($E = +\sqrt{p^2c^2 + m^2c^4}$) there existed a negative-energy solution ($E = -\sqrt{p^2c^2 + m^2c^4}$).This meant that due to the natural tendency of every system to remain in lower energy states an electron might rush towards negative states emitting an infinite amount of energy but obviously this is not the case.Dirac proposed a model to resolve this problem.He postulated that the negative energy states are all occupied by an infinite sea of electrons.Because this sea is perfectly uniform it doesn't interact with anything so we are unable to observe the particles of that sea. If sufficient energy is provided to an electron in that sea it would lead to the absence of -ve charge and -ve energy at that position or the presence of a particle with +ve charge and +ve energy.Such a particle named 'positron',a positively charged twin for the electron was discovered in 1931 by Anderson.

1.3.1 Notations for Antiparticles

The standard notation for antiparticles is an **overbar**.For eg p $\rightarrow$ proton; $\bar{p} \rightarrow antiproton$.But in cases of charged particles usually the charge is specified on the particle.For eg $e^+ \rightarrow positron$;$\mu^+ \rightarrow antimuon$.Some neutral particles are their own antiparticles eg. the photon : $\bar{\gamma} = \gamma$.

1.3.2 Crossing symmetry

According to the principle of crossing symmetry ,if a reaction

$$A + B \rightarrow C + D$$

is known to occour then any of the particles involved in the reaction can be "crossed" over to the other side of the equation ,provided it is turned into its antiparticle,and the resulting reaction will also be allowed.For eg.

$$A \rightarrow \bar{B} + C + D$$

$$A + \bar{C} \rightarrow \bar{B} + D$$
$$\bar{C} + \bar{D} \rightarrow \bar{A} + \bar{B}$$

Note that although a reaction may be dynamically allowed due to the principle of crossing symmetry it can still be kinematically disallowed.

1.4 Neutrinos

The search for neutrinos began when extensive studies of beta decay were carried out in 1930's.In beta decay a radioactive nucleus A is transformed into another radioactive nucleus B with the emission of an e^-

$$A \rightarrow B + e^-$$

For such two particle decays like above simple relativistic energy and momentum conservation equations calculate that the energy of outcoming electron is given by the equation

$$E_e = \left(\frac{m_A^2 - m_B^2 + m_e^2}{2m_A} \right) c^2$$

But experiments showed that instead this energy E_e is the **maximum** electron energy for a beta decay process.To resolve this problem Pauli proposed the emission of another light, neutral particle in beta decay ,which carries away a huge fraction of E_e, and named it neutron(which was later changed to neutrino).Similar observations were also made for pion and muon decays.So the correct equation for the three decay processes are as follows

$$n \rightarrow p^+ + e^- + \bar{\nu}$$
$$\pi \rightarrow \mu + \bar{\nu}$$
$$\mu \rightarrow e + 2\nu$$

By 1950 there was a clear theoretical evidence for the existence of neutrinos but they were not yet found experimentally.Cowan and Reines conducted "inverse" beta-decay type of reaction experiments

$$\bar{\nu} + p^+ \rightarrow n + e^+$$

The results of their experiments provided an unambiguous confirmation of existence of neutrinos.Another question that was posed was since neutrinos are neutral is there any distinction between neutrino and antineutrino.This problem was answered by the experiments conducted by Harmer and Davis who looked for a reaction analogus to

$$\nu + n \rightarrow p^+ + e^-$$

(from positive test results of Cowan and Reiness,above cross symetric reaction using neutrinos was sure to occour) using antineutrinos:

$$\bar{\nu} + n \rightarrow p^+ + e^-$$

But they found that this reaction does not occour,hence it was proved that antineutrino and neutrino are two distinct particles.

1.5 Strange Particles

In 1947 Rochester and Butler,while conducting some analysis of cosmic rays discovered a neutral particle which decays into π^+ and π^-.This particle was named Kaon(K^0).Two years later Powell found the decay of a charged Kaon into π^+, π^+, π^-.The Kaons behave as heavy pions so were included in meson family. In 1950 another neutral particle Λ was discovered whose decay products were p^+ and π^-.Λ must be grouped with neutrons and protons in baryon family if 'baryon number conservation'(Refer section 2) were to hold true. Similary other baryons like Σ, Ξ, Δ were found later.All the new heavy baryons and mesons were collectively called 'strange' particles.Their production was fast(around 10^{-23}seconds) but their decay was comparatively slow(around 10^{-10}seconds).A new quantity called 'strangeness' was assigned to each particle,which was conserved in any strong interaction but not in weak interaction.K's were assigned strangeness S=+1,the $\Sigma's$ and the $\Lambda's$ have S=-1 and the ordinary particles - π,p,n have S=0.
For eg. production of strange particles always take place in pairs and we do not encounter a single strange particle produced in a reaction.

$$\pi^- + p^+ \rightarrow K^+ + \Sigma^-$$
$$\pi^- + p^+ \nrightarrow \pi^+ + \Sigma^-$$

S=0 for both sides in first reaction while S=0 for LHS and S=-1 for RHS,hence second reaction is not possible.
Since decay of strange particles proceed through weak interaction strangeness is not conserved in that case.

$$\Lambda \rightarrow p^{+} + \pi^{-}$$

1.6 The Eightfold Way

The Eightfold Way,first proposed by Murray Gell-Mann was the arrangement of baryons and mesons into different geometrical patterns,according to their charge and strangeness.The eight lightest baryons fit into a hexagonal array with two particles at the center forming the baryon octet.Similarly eight lightest mesons were arranged in a hexagonal pattern to form the meson octet which later become meson nonet after discovery of η'.

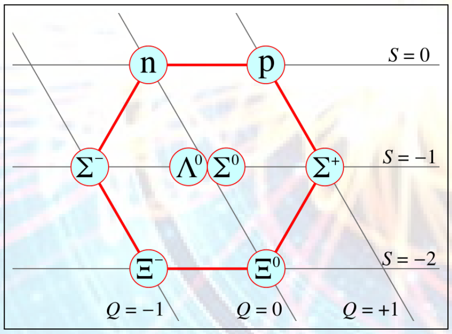

Figure 1.6.1: The Baryon Octet

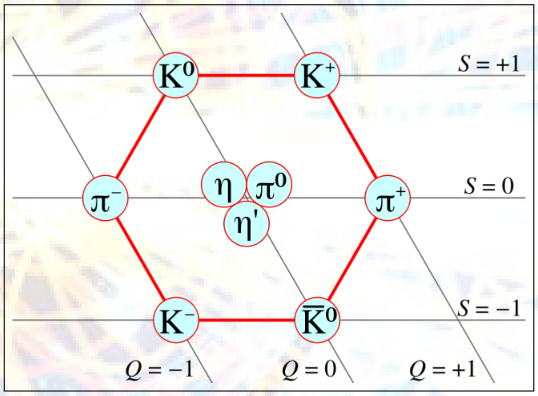

Figure 1.6.2: The Meson Nonet

10 heavier baryons were arranged in triangular arrays known as baryon decuplet.Similar octets and decuplets exist for antibaryons as well. In case of mesons the antiparticles lie in the supermultiplet as the corresponding particles,in the diametrically opposite directions.For example $\pi^{+} - \pi^{-}$ are a particle-antiparticle pair.

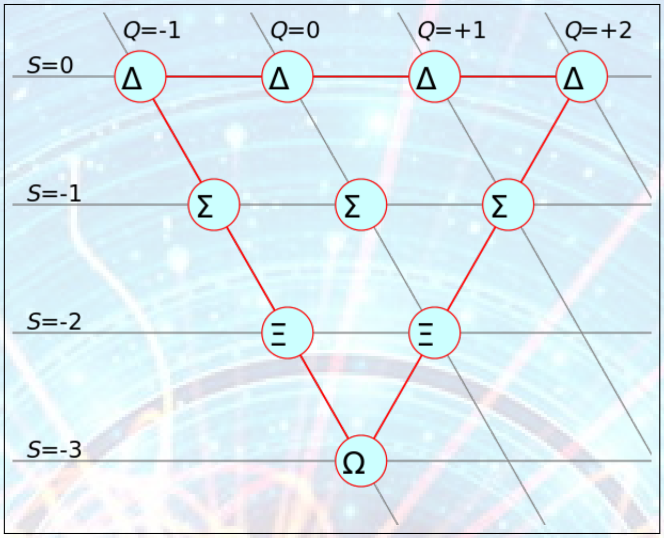

Figure 1.6.3: The Baryon Decuplet

1.7 The Quark Model

To explain why do the hadrons fit into these curious patterns Gell-Mann and Zweig indpendently proposed that all hadrons are in fact composed of even more elementary particles known as quarks.The quarks come in three type(”flavors”) forming a triangular ”Eightfold Way” pattern

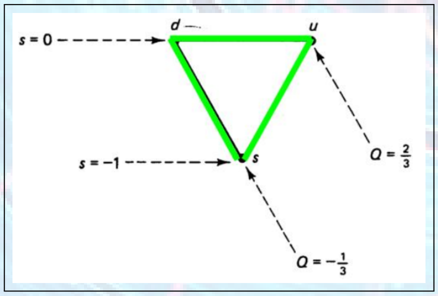

Figure 1.7.1: The Quark Model

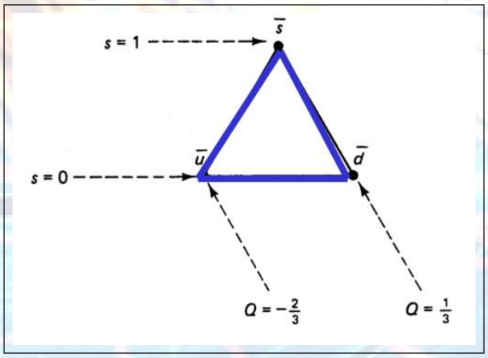

Figure 1.7.2: Anti-quarks are just opposite in charge and strangeness

Each baryon is composed of three quarks and each meson is composed of a quark and an antiquark.We can combine the three quarks in 10 different ways to obtain the baryon decuplet and a quark and antiquark in 9 different ways to obtain the meson nonet as shown.The last particle in meson nonet is η' which was later added to the meson octet.

qqq	Q	S	Baryon
uuu	2	0	Δ^{++}
uud	1	0	Δ^{+}
udd	0	0	Δ^{0}
ddd	−1	0	Δ^{-}
uus	1	−1	Σ^{*+}
uds	0	−1	Σ^{*0}
dds	−1	−1	Σ^{*-}
uss	0	−2	Ξ^{*0}
dss	−1	−2	Ξ^{*-}
sss	−1	−3	Ω^{-}

Figure 1.7.3: The Baryon Decuplet

$q\bar{q}$	Q	S	Meson
$u\bar{u}$	0	0	π^0
$u\bar{d}$	1	0	π^+
$d\bar{u}$	−1	0	π^-
$d\bar{d}$	0	0	η
$u\bar{s}$	1	1	K^+
$d\bar{s}$	0	1	K^0
$s\bar{u}$	−1	−1	K^-
$s\bar{d}$	0	−1	$\bar{K}^0$
$s\bar{s}$	0	0	??

Figure 1.7.4: The Meson Nonet

One important thing to note here is that the same combination of quarks can bind together to give different particles.Eg. π^+, ρ^+ are both $u\bar{d}$ and p^+, Δ^+ are both uud.
Just as in a hydrogen atom the electron-proton system has different energy levels in the same way a given collection of quarks can bind together to produce several combinations at different energy levels.But the energy levels spacing in the quarks system is quite high so we consider different states of a system of given quarks as individual particles.

In spite of the huge success of the quark model ,it had some problems.First one being that no one has ever been able to observe individual quark.The phenomena of color confinement is supposed to be an explanation for this problem which is discussed in section 2.2.2.
The second problem was that the model appeared to violate the Pauli's Exclusion Principle according to which no two(or more) particles with half integer spin can occupy same quantum state within a quantum system.Since

quarks are also half-integer spin paticle(we'll discuss later how) Pauli's principle also apply to them.W. Greenberg proposed a solution of this dilemma in 1964 by suggesting that quarks not only come in three flavor(u,d or s) but each one of those also come in three colors ("red", "green", "blue").A baryon consists of three quarks one of each color.For eg there are three u's in Δ^{++} one of them being red, the other one being green, and the remaining oneblue.Since the three quarks are no more identical the dilemma is resolved.

1.8 The November Revolution

In the November of 1974 the discovery of a new particle ψ meson was reported.It had very bizarre properties like an unexpectedly longer lifetime of $10^{-20}s$ as compared to other $10^{-23}s$ lifetime of other particles in similar mass range.Subsequent studies of this meson revealed that ψ represents bound state of a new quark ,the c (for **charm**) and its antiquark ($\psi = c\bar{c}$). The existence of this fourth quark had been predicted by Glashow and Bjorken much earlier before its discovery owing to the fact that there were some similarities between quarks and leptons.And since both were true fundamental particles there numbers could also be equal.In 1975 a new lepton (τ) was discovered and subsequently its corresponding neutrino.This lead to breaking of Glashow'g symmetry which was soon restored after the discovery of two new quarks viz beauty/bottom(b) and truth/top(t). So now we have 6 quarks and 6 leptons in total.

1.9 Intermediate Vector Bosons

In the original beta decay hypothesis of Fermi he treated the process as a contact interaction occuring at a single point and therefore requiring no mediating particle.Since weak interactions are very short range forces this model gave approximately correct results at lower energies.However it was later found that at higher energies this model would violate and therefore a need for mediating particles of weak forces was felt.These mediating particles were named intermediate vector bosons initially and they were 3 in number - two charged(W^+, W^-) and one uncharged(Z)and their masses were predicted to be around 100 times that of proton.Indeed these heavy mediating particles were observed in CERN proton-antiproton collider.

1.10 Putting it all together-The Standard Model

Let us sumarrize this discussion of particles and see what all we've got.First we have leptons.There are 3 generations of leptons(e^-, μ^-, τ^-) and each generation corresponds to the particle and its neutrino.Each lepton has a corresponding anti-lepton as well.So in total there are 12 leptons(or leptons+antileptons) .

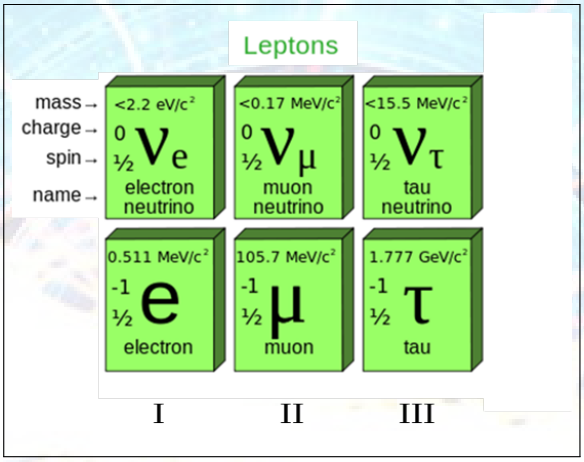

Figure 1.10.1: The three lepton generations

Now coming on to quarks,just like leptons there are 3 quark generations each comprising of 2 different flavored quarks(making in total 6 different flavors).Now each quark also comes in three colors which makes there number 18 and each quark has a corresponding anti-quark as well so shooting the quark number to 36 in total.

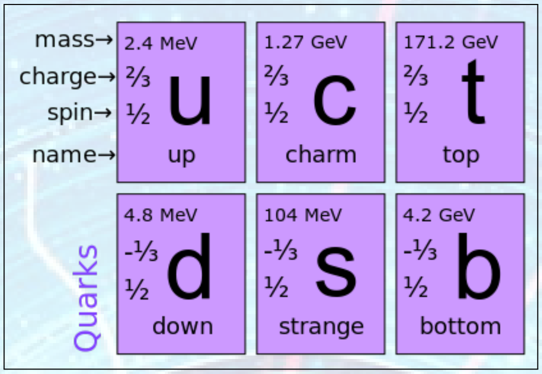

Figure 1.10.2: The three quark generations

Now we discuss about mediating particles.As we know EM force has photon(γ),and weak force has three mediating particles(W^+, W^-, Z).Strong force also has mediating particles named gluons which are 8 in number about which we'll study in great detail in subsequent chapters.So in total there are 12 mediating particles.Also there is one another particle known as Higg's particle.So in total there are 61 particles.Leptons and Quarks are classified together as Fermions while the mediating particles along with Higgs particle are known as Bosons.These Bosons and Fermions together constitute what is known as "**The Standard Model of Particle Physics**".

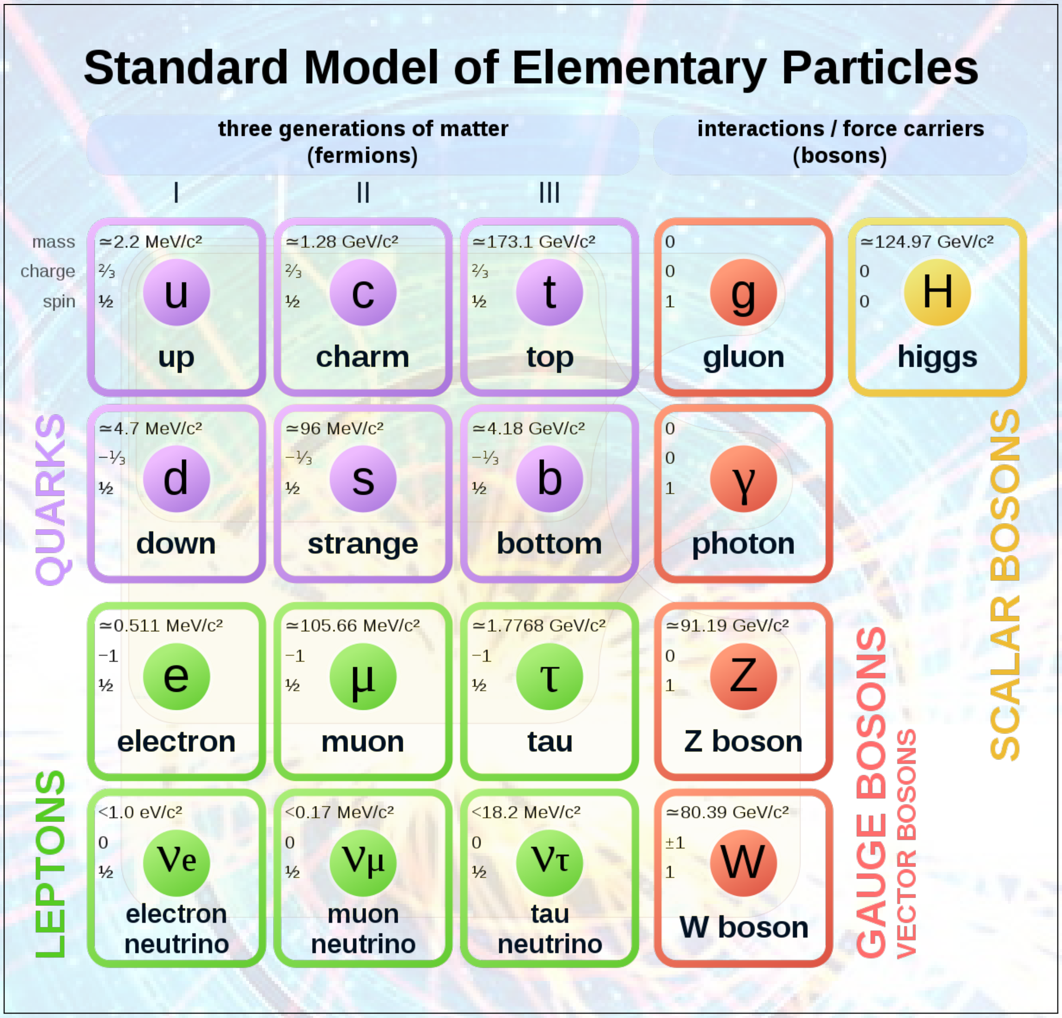

Figure 1.10.3: The Standard Model

2 Elementary Particle Dynamics

Upto our current understanding of the universe all types of forces in nature can be classified in the following four types
Note: The strength measures mentioned here are just representative of

Force	Strength	Theory	Mediator
Strong	10	Chromodynamics	Gluon
Electromagnetic	10^{-2}	Electrodynamics	Photon
Weak	10^{-13}	Flavordynamics	W and Z
Gravitational	10^{-42}	Geometrodynamics	Graviton

their relative strengths and major fluctuations can be observed in some cases(especially in Weak Forces).

As evident from the table each type of force has a physical theory belonging to it.We start looking at them one by one starting with Electrodynamics.

2.1 Quantum Electrodynamics (QED)

Of the four theories available QED is the oldest and the most succesful one.Other theories are modelled on it.We start our discussion of QED by introducing the notion of Feynman Diagrams.
One of the most basic feynman diagram that can be drawn is that for a repulsive interaction between two electrons also known as Moller Scattering.

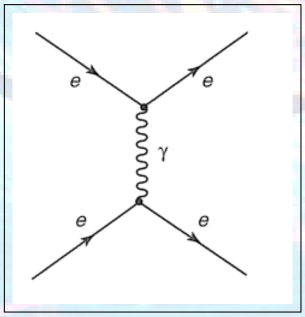

Figure 2.1.1: Moller Scattering

The diagram represents interaction of two electrons mediated by a photon.

Note: In all the diagrams I will use the convention that time flows from **left to right** ,though some books might use the other convention in which time is assumed to be flowing from bottom to up.
Also all the particles whose arrowheads point in the forward direction represent the mentioned particles while the particles with arrowheads pointing in the backward direction represent the particle going backwards in time and is interpreted as the corresponding antiparticle going forward in time.

Similarly we can draw following diagrams for interaction of e^- and e^+

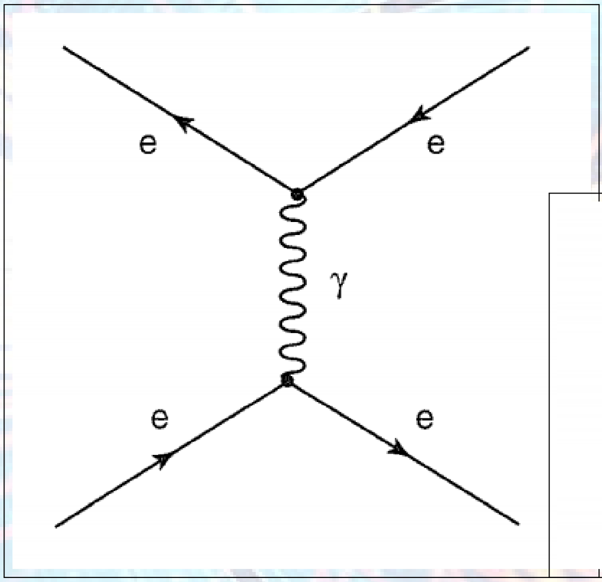

Figure 2.1.2: Bhabha Scattering

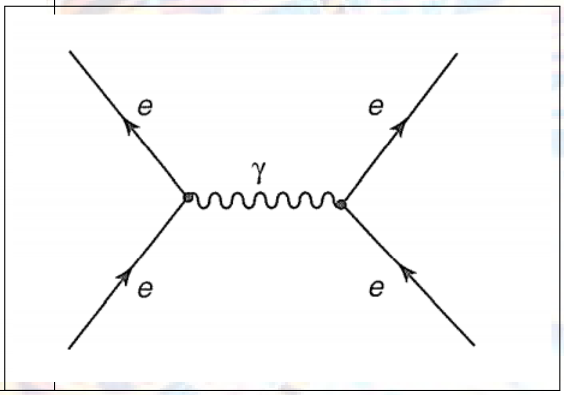

Figure 2.1.3: Bhabha Scattering(Another Possible Diagram)

2.1.1 Fine Structure Constant

For a physical process an infinite number of Feynman diagrams can be drawn consisting of an infinite number of vertices.But each vertex introduces a coupling constant of $\alpha = e^2/\hbar c$ (the fine structure constant).So as the number of vertices increases the contribution keeps on decreasing and usually diagrams having more than four vertices do not contribute to the final calculations.

2.2 Quantum Chromodynamics(QCD)

In chromodynamics color plays the role of charge and the fundamental process is quark → quark + gluon.Since leptons do not carry color they do not take part in strong interactions.Unlike one kind of electric charge(which can either be +ve or -ve) in QED,there are three types of colors(red,green,blue)in QCD and unlike electrically neutral photon ,gluons carry color.Since at each vertex color conservation holds true ,the emission of a bicolored gluon would simply mean that in a strong interaction color of a quark might change however its flavor remains the same.Directing gluon-gluon coupling is also possible in QCD due to the assosciation of color with a gluon.

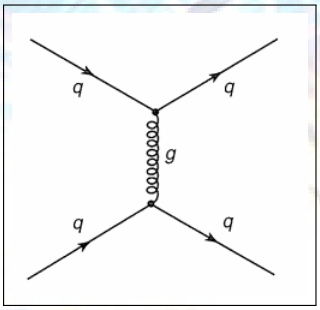

Figure 2.2.1: Interaction beyween two quarks mediated by a ”gluon”

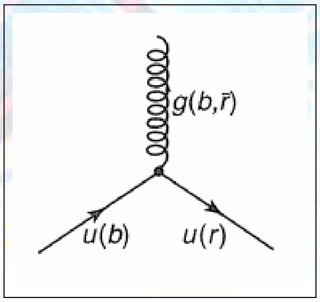

Figure 2.2.2: Emission of a ’colorless’ gluon

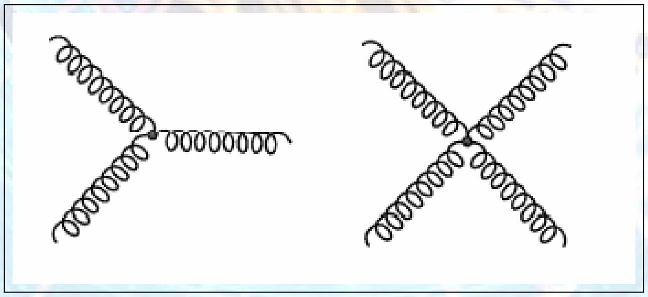

Figure 2.2.3: gluon-gluon coupling

2.2.1 Asymptotic Freedom

In QCD the coupling constant (similar to fine structure constant in QED) was found to be greater than 1 initially which could have lead to some serious problems as the contribution of diagrams with larger number of vertices would have increased in stead of decresing but it was later found that the value of this coupling constant was not at all constant and was dependent on the distance between the interacting particles.At large distances the value of coupling constant was large but at quite small distances (high energy regime eg. inside a proton) the value was quite small.This property is known as asymptotic freedom.The nature of coupling can be found out by looking at the parameter $\alpha = 2f - 11n$ where f is no. of flavors and n is no. of colors.In standard model f=6 and n=3 so $\alpha = -21$(negative) hence the coupling decreases at shorter distances.

2.2.2 Color Confinement

This property of color charged particles(quarks and gluons) to always occur in colorless combinations is color confinement.

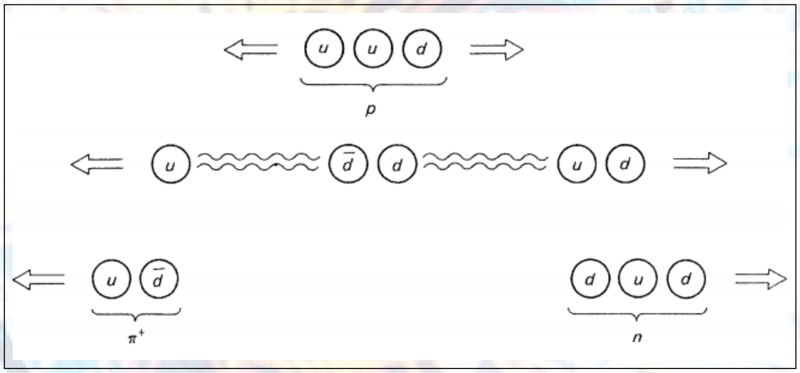

Figure 2.2.4: color confinement

2.3 Quantum Flavorodynamics

There are two types of weak interactions: charged(mediated by W bosons) and neutral (mediated by Z bosons).Both leptons and quarks take part in weak interactions

2.3.1 Neutral Weak Interactions

The most fundamental neutral vertex can be drawn as shown in the following diagram.Some common examples of neutral interactions are ν_μ-e scattering or ν_μ-p sattering.Note that any process mediated by the photon can also be mediated by Z but Weak interactions being much weaker than Electromagnetic interactions makes their detection very difficult in such processes.

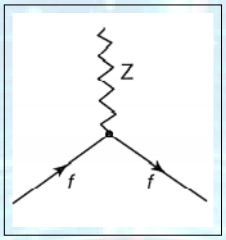

Figure 2.3.1: f is any quark or lepton

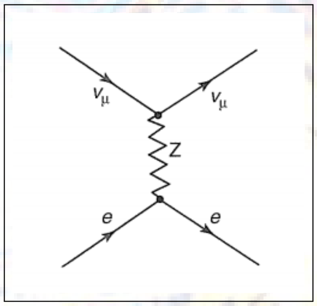

Figure 2.3.2: ν_μ-e scattering

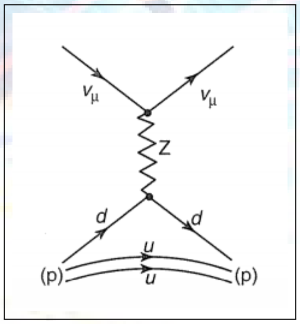

Figure 2.3.3: ν_μ-p scattering

2.3.2 Charged Weak Interactions-Leptons

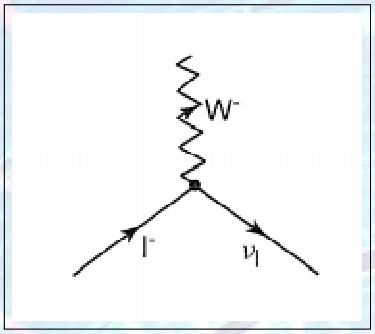

Figure 2.3.4: Fundamental vertex in case of leptonic charged weak interaction (l is a negative lepton and ν_l is corresponding neutrino)

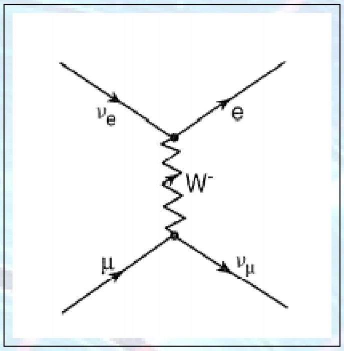

Figure 2.3.5: $\mu^- + \nu_e \rightarrow e^- + \nu_\mu$

2.3.3 Charged Weak Interactions- Quarks

At a leptonic charged weak vertex the lepton generation doesn't change i.e. e^-, ν_e;μ^-, ν_μ;τ^-, ν_τalways occur in pairs at each vertex.
But in case of quarks this picture is a bit more complicated.First we look at the simpler 'generation-preserving' interactions.The fundamental vertex in this case looks like as shown below.Note that the color is conserved for this vertex but flavor changes

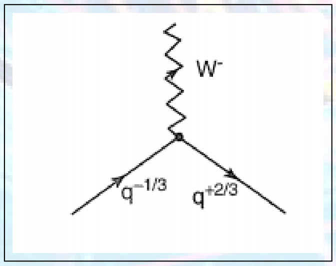

Figure 2.3.6: $q^{-1/3}$:d,s,b;$q^{2/3}$:u,c,t

The other end of this vertex can couple to leptons(a semi-leptonic process) or to other quarks(a purely hadronic process).

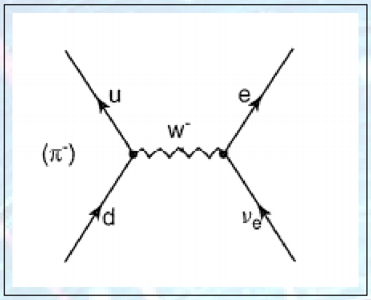

Figure 2.3.7: $\pi^- \rightarrow e^- + \bar{\nu}_e$

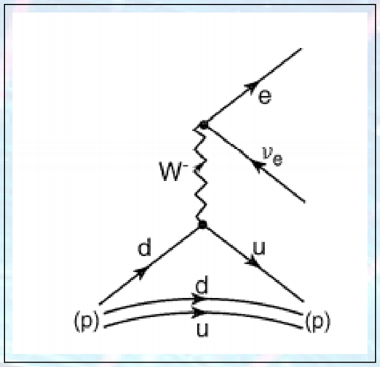

Figure 2.3.8: $n \rightarrow p^+ e^- + \bar{\nu}_e$

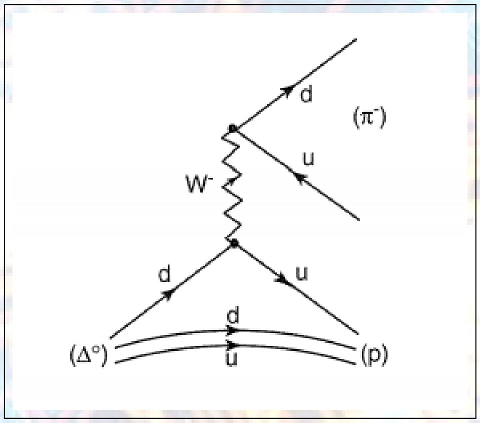

Figure 2.3.9: Replacing $e^- - \nu_e$ vertex in last example by quark vertex gives us a purely hadronic interaction

One imprtant thing to note is that the decay $\Delta^0 \rightarrow p^+ + \pi^-$ can also proceed through strong interaction and due to very small contribution from Weak Mechanism makes its detection quite difficult.

Now we address the 'complications' referred earlier in case of charged weak interactions in quarks.There are some interactions like the decay of lambda ($\Lambda \rightarrow p^+ + \pi^-$) or omega-minus ($\Omega^- \rightarrow \Lambda + K^-$) which can only be explained if we allow the quark generation to change(for eg strange quark to up quark in this case)

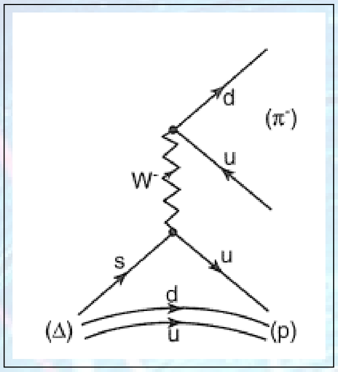

Figure 2.3.10: Λ decay

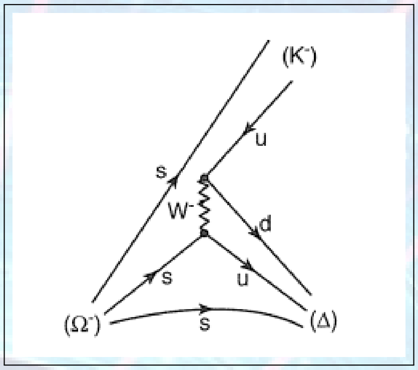

Figure 2.3.11: $\Omega^{-}decay$

This inter generation conversion among quarks was explained by using the argument that instead of $\begin{bmatrix} u \\ d \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} \begin{bmatrix} t \\ b \end{bmatrix}$ the weak forces couples the pairs $\begin{bmatrix} u \\ d' \end{bmatrix} \begin{bmatrix} c \\ s' \end{bmatrix} \begin{bmatrix} t \\ b' \end{bmatrix}$

where d', s', b' are linear combinations of physical quarks d,s,b related by a 3×3 *Kobayashi-Maskawa* matrix as follows

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \cdot \begin{bmatrix} d \\ s \\ b \end{bmatrix} \approx \begin{bmatrix} 0.974 & 0.227 & 0.004 \\ 0.227 & 0.973 & 0.042 \\ 0.008 & 0.042 & 0.999 \end{bmatrix} \cdot \begin{bmatrix} d \\ s \\ b \end{bmatrix}$$

Here V_{ud}measures the coupling of u to d, V_{us} measures the coupling of u to s and so on.

2.4 Conservation Laws

All particle decays can be classified in the following three types based on the fundamental forces that are involved

Type of decay	Meanlife
Strong	$10^{-23}sec$
Electromagnetic	$10^{-16}sec$
Weak	$10^{-13}sec - 15min$

Conservation laws allow us to predict whether a certain decay is possible or not.Other than kinematic conservation laws of conservation of Energy,Momentum and Angular Momentum there are certain dynamical conservation laws that are based on the structure of fundamental vertices that we have discussed.

1. **Charge** : Charge is conserved in all three interactions.
2. **Color**:EM and weak interactions do not affect color.At strong vertex the color is conserved at both quark-gluon and gluon-gluon vertex.
3. **Baryon Number**:In every primitive vertex quark number is always conserved but since due to color confinement individual quarks are never observed so we keep our discussion confined to Baryons(quark no. 3),Antibaryons(quark no. -3) and mesons(quark no. 0).Accordingly we assosciate Baryon number(A=1 for Baryons,A=-1 for Antibaryons) and talk about conservation of Baryon number.However no such analogus Meson number conservation holds for Mesons owing to the fact that they have total quark number equal to 0.
4. **Lepton Number**: Leptons are unaffected by strong forces.In electromagnetic interactions same lepton comes out which goes in and in weak interaction also if a lepton goes in ,a lepton comes out (though it's nature might change).Infact for the three leptonic generations conservation of electron,muon and tau number also holds separately.(Though neutrino oscillations might be one possible exception)
5. **Flavor**: Flavor is conserved at a strong or electromagnetic vertex but not at weak vertex(for uncharged weak interactions the flavor conservation holds)as in case of charged weak interactions an up quark can convert into a down or strange quark by emission of a W^- boson.

2.5 Towards Grand Unification

Physicists have been proposing various schemes to bring all the fundamental forces in nature under one roof a single **"Grand Unified Theory(GUT)"**.Till now they've been successful in unification electromagnetic and weak force known as 'Electroweak force' and unifying it with strong force to a large extent.

In our discussion of coupling constants we saw that for electromagnetic and strong forces the value of coupling constant increases and decreases respectively in high energy regime.For weak forces the value of coupling constant decreses but at a smaller rate .It is proposed that at very high energy scales($\approx 10^{15}$ Gev) the three coupling constants attain a common value as shown below.

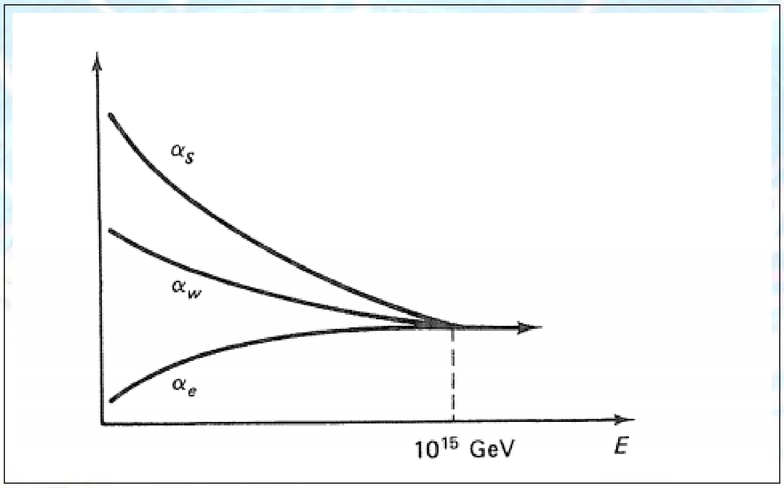

Figure 2.5.1: Variation of the three coupling constants with Energy

Another prediction of this GUT model is the prediction of some reactions like $p^+ \to e^+ + \pi^0$ or $p^+ \to \bar{\nu_\mu} + \pi^+$ where Baryon number conservation does not hold.However no such proton decays have been observed as of now.
If this unification of strong force with electroweak force works out completely then Gravity would be the only force left out of this unification model.Several theories like String Theory,Quantum Gravity have been proposed to achieve this miraculous task of ultimae grand unification,but none have yet proven to be completely accurate.

3 Matrix Mechanics

We are aware that the quantity 'spin' has no classical analog and therefore it becomes difficult to write vectors, wavefunctions and operators in real space.Rather we tend to express wavefunctions and operators in an **abstract** space.This way of expressing operators and wavefunctions is known as **Matrix Mechanics**.

The basic idea is that we can write any electron spin state as a linear combination of the two states α and β:

$$\psi = c_\alpha \alpha + c_\beta \beta \tag{3.1}$$

where $\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents vectors corresponding to 'spin-up' and 'spin-down' states respectively in an abstract vector space.

Equation 3.1 implies that $\mathbf{\Psi} = \begin{bmatrix} c_\alpha \\ c_\beta \end{bmatrix}$ is a vector in the same abstract vector space.

Now we try to transform every operation of wavefunction ψ into an operation of vector $\mathbf{\Psi}$.

1. Integrals replaced with dot products The overlap between any two wavefunctions can be written as a modified dot product between two vectors.Eg consider $\phi = d_\alpha \alpha + d_\beta \beta$.Then it can be easily shown that

$$\begin{aligned} \int \phi^* \psi d\tau &= \begin{bmatrix} d_\alpha^* & d_\beta^* \end{bmatrix} \cdot \begin{bmatrix} c_\alpha \\ c_\beta \end{bmatrix} \\ &= \mathbf{\Phi}^\dagger \cdot \mathbf{\Psi} \end{aligned} \tag{3.2}$$

2. **Complex conjugation of the wavefunction is replaced by taking the adjoint of a vector**-Can be concluded directly from equation 3.2

3. **Normalization and Orthogonalization**

$$\int \psi^*.\psi d\tau = 1$$
$$\implies \Psi^\dagger.\Psi = 1$$
$$\implies |c_\alpha^2| + |c_\beta^2| = 1 \tag{3.3}$$

$$\int \phi^*.\psi d\tau = 0$$
$$\implies \Phi^\dagger.\Psi = 0$$
$$\implies d_\alpha^*.c_\alpha + d_\beta^*.c_\beta = 0 \tag{3.4}$$

4. **Operators are represented by matrices**-For eg.the $\hat{S}_z$ operator will be transformed into a 2×2 matrix in spin space

$$\hat{S}_z \longrightarrow \boldsymbol{S_z} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Using $\hat{S}_z\alpha = \frac{\hbar}{2}\alpha$ and $\hat{S}_z\beta = -\frac{\hbar}{2}\beta$,we can obtain

$$\boldsymbol{S_z} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.5}$$

4 Fundamentals of Group Theory

4.1 Symmetry

Symmetry is an operation which we can perform on a system that leaves it invariant. In 1917 Emmy Noether published her famous theorem relating symetries and conservation laws.Every symmetry of nature yields a conservation law.Typically there are 4 types of symmetries that we deal with in our day to day life which are given in the following table along with corresponding conservation law according to Noether's Theorem.

Symmetry		Conservation Law
Translation in Time	$\Longleftrightarrow$	Energy
Translation in Space	$\Longleftrightarrow$	Momentum
Rotation	$\Longleftrightarrow$	Angular Momentum
Gauge Transformation	$\Longleftrightarrow$	Charge

The set of all symmetry operations (on a particular system)has the following properties.

1. **Closure:** If R_i and R_j are in the set then the product R_iR_j is also in the set i.e. $R_k = R_iR_j$ for some R_k in the set
2. **Identity:**There is an element I such that $IR_i = R_i$ for all elemnts R_i
3. **Inverse:**For every elemnet R_i there is an invere, R_i^{-1} such that $R_iR_i^{-1} = R_i^{-1}R_i = I$.
4. **Assosciativity:**$R_i(R_jR_k) = (R_iR_j)R_k$

These are the defining properties of a mathematical **group**.Group theory is the systematic study of symetries.Groups can be *finite* (like the triangle group which has 6 elements of symmetry) or *infinite*(like the group of integers).We will be covering continuous groups (eg. group of all rotations in a plane) in a great detail.

4.2 Continuous(Lie) Groups

4.2.1 Introduction

For particle physics it is most natural to introduce the ideas of symmetry in the context of quantum mechanics. In quantum mechanics, a symmetry of the Universe can be expressed by requiring that all physical predictions are invariant under the wavefunction transformation.

$$\psi \to \psi' = \hat{U}\psi$$

$\hat{U}$ is the operator corresponding to a particular symmetry transformation.The requirement that all physical predictions are unchanged by a symmetry transformation, gives to rise some constraints on $\hat{U}$.
Invariance of wavefunction normalisation implies

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle$$
$$\hat{U}^\dagger\hat{U} = I \tag{4.1}$$

Here I represents unity,which could be 1 or identity matrix(depending upon whether ψ is a function or a vector).

Furthermore, for physical predications to be unchanged by a symmetry operation, the eigenstates of the system also must be unchanged by the transformation. Hence the Hamiltonian itself must possess the symmetry in question, $\hat{H} \to \hat{H}' = \hat{H}$

The eigen states of the Hamiltonian satisfy $\hat{H}\psi_i = E_i\psi_i$,and because of the invariance of the Hamiltonian, the energies of the transformed eigenstates ψ_i' will be unchanged.

$$\hat{H}'\psi' = \hat{H}\psi_i' = E_i\psi_i'$$
$$\hat{H}\hat{U}\psi_i = E_i\hat{U}\psi_i = \hat{U}E_i\psi_i = \hat{U}\hat{H}\psi_i$$
$$\implies [\hat{H},\hat{U}] = \hat{H}\hat{U} - \hat{U}\hat{H} = 0 \tag{4.2}$$

Hence, for each symmetry of the Hamiltonian there is a corresponding unitary operator which commutes with the Hamiltonian.

Now we consider a case of 1-D translational symmetry to introduce the concept of **generators**. Consider a free particle of mass m having a momentum p_x in x-direction.For the particle,

$$H = \frac{p^2}{2m}$$
$$p = \frac{dx}{dt}$$

Now let the particle undergo a translation ,

$$x \longrightarrow x' = x + a_x$$

The particle's Hamiltonian and wavefunction change as follows

$$H \longrightarrow H' = H$$
$$\psi(x) \longrightarrow \psi'(x) = \hat{U}(a_x)\psi(x) = \psi(x + a_x) \tag{4.3}$$

where $\hat{U}(a_x)$ is the translational operator which when operated on wavefunction(or any arbitary function)$\psi(x)$ generates the new wavefunction $\psi'(x)$. We can use Taylor expansion of $\psi(x+a_x)$ about x to obtain the translational operator $\hat{U}(a_x)$.

$$1\psi(x + a_x) = \psi(x) + a_x \frac{\partial \psi(x')}{\partial x}\bigg|_{x'=x} + \frac{a_x^2}{2!}\frac{\partial^2 \psi(x')}{\partial x^2}\bigg|_{x'=x} + \cdots \tag{4.4}$$
$$= \left[1 + a_x \frac{\partial}{\partial x} + \frac{a_x^2}{2!}\frac{\partial^2}{\partial x^2} + \cdots\right] \psi(x')\bigg|_{x'=x} \tag{4.5}$$
$$= e^{a_x \frac{\partial}{\partial x}} \psi(x')\bigg|_{x'=x} \tag{4.6}$$

Thus

$$\boxed{\hat{U}(a_x) = exp\left[a\frac{\partial}{\partial x}\right]} \tag{4.7}$$

In case of infinitesimal translation δa_x

$$\psi'(x) = \psi(x + a) = \left[1 + \delta a_x \frac{\partial}{\partial x}\right] \psi(x) + \mathcal{O}(\delta^2(a_x)) \tag{4.8}$$

For this infinitesimal transformation terms of $\mathcal{O}(\delta^2(a_x)$ can be neglected.using this approximation any finite transformation can be expressed as a series of large number of infintesimal transformations as follows

$$a = N(\delta a)$$
$$\psi(x+a) = \lim_{N\to\infty}\left[1 + \frac{a\partial}{N\partial x}\right]^N \psi(x) \tag{4.9}$$

So we tend to study only infintesimal transformations.
In case of infintesimal transformations,generally we express the unitary operator $\hat{U}$ as,

$$\hat{U}(\epsilon) = I + \iota\epsilon\hat{G} \tag{4.10}$$

where ϵ is an infinitesimally small parameter and $\hat{G}$ is called the **generator** of the transformation. Since is $\hat{U}$ unitary

$$\hat{U}(\epsilon)\hat{U}^{\dagger}(\epsilon) = I = (I + \iota\epsilon\hat{G})(I - \iota\epsilon(\hat{G}^{\dagger})) \tag{4.11}$$
$$I = I + \iota\epsilon(\hat{G} - \hat{G}^{\dagger}) \tag{4.12}$$

$$\implies \boxed{\hat{G} = \hat{G}^{\dagger}} \tag{4.13}$$

Thus, for each symmetry of the Hamiltonian there is a corresponding unitary symmetry operation with an associated Hermitian generator $\hat{G}$. The eigenstates of a Hermitian operator are real and therefore the operator $\hat{G}$ is associated with an observable quantity G. Furthermore, since $\hat{U}$ commutes with the Hamiltonian, $[\hat{H}, I + \iota\epsilon\hat{G}] = 0$, the generator $\hat{G}$ also must commute with the Hamiltonian

$$[\hat{H}, \hat{G}] = 0 \tag{4.14}$$

The time evolution of the expectation value of the operator $\hat{G}$ is given by

$$\frac{d}{dt}\langle\hat{G}\rangle = \iota\langle[\hat{H}, \hat{G}] \tag{4.15}$$

From eq. 4.14,

$$\boxed{\frac{d}{dt}\langle\hat{G}\rangle = 0} \tag{4.16}$$

Hence, for each symmetry of the Hamiltonian, there is an associated observable conserved quantity G which is the expression of Noether's theorem.

Now we'll return back to our example of 1D translation to analyse which observable quantity is assosciated with its generator but befor we do that ,we observe that unitary operator for 3D translation can be constructed just like in 1D case.
It can be easily verified that

$$\boxed{\hat{U}(\boldsymbol{a}) = exp(\boldsymbol{a}.\boldsymbol{\nabla}) = exp(a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z})} \tag{4.17}$$

where $\boldsymbol{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ is the translation in 3d coordinates. Now we try to write eq 4.8 using the definition of linera momentum operator$(-\iota\hbar\frac{\partial}{\partial x})$,

$$\psi(x+a) = \left[1 + \delta a_x.(\frac{\iota\hat{p_x}}{\hbar})\right]\psi(x) \tag{4.18}$$

Comparing with eq 4.10 we can conclude that generator $\hat{G}$ for 1D translation operator $\hat{U}(a_x)$ is the linear momentum operator in 1D $\hat{p_x}$.
This result can be further extended to 3D translation.But unlike 1D case here we would get a set of three generators $\hat{p_x}, \hat{p_y}, \hat{p_z}$ corresponding to 3 infintesimal parameters $(\epsilon_x, \epsilon_y, \epsilon_z)$.
In general, a symmetry operation may depend on more than one parameter, and the corresponding infinitesimal unitary operator can be written in terms of the set of generators $\hat{\boldsymbol{G}} = \{\hat{G}_i\}$

$$\hat{U} = 1 + \iota\boldsymbol{\epsilon}.\hat{\boldsymbol{G}} \tag{4.19}$$

where $\boldsymbol{\epsilon} = \{\epsilon_i\}$

From the result of eq 4.16 we can easily conclude that the quantity which remains conserved in case of translational symmetry is linear momentum.

4.2.2 Rotational symmetry and SO(3) groups

Again we consider a particle undergoing a rotation,

$$\boldsymbol{r} \longrightarrow \boldsymbol{r'} = R(\psi, \hat{n})\boldsymbol{r}$$

Here ψ is the magnitude of angle of rotation and $\hat{n}$ is unit vector along the axis of rotation.
For simplicity let us consider rotation along $\hat{z}$.One thing to note here is that from now we'll use matrix mechanics for representation of operators and wavefunctions.

$$R(\delta\psi,\hat{z}) = \begin{bmatrix} cos(\delta\psi) & sin(\delta\psi) & 0 \\ -sin(\delta\psi) & cos(\delta\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .\delta\psi \tag{4.20}$$

Also we observe that

$$\boldsymbol{r'} = \boldsymbol{r} + \boldsymbol{\delta r} \tag{4.21}$$

where $\boldsymbol{\delta r} = \delta\psi_y\hat{x} - \delta\psi_x\hat{y}$ for rotation along z-axis.

$$\begin{aligned} \boldsymbol{\delta r_z} &= \boldsymbol{\delta\psi_z} \times \boldsymbol{r} \\ \boldsymbol{\delta r_R} &= \boldsymbol{\delta\psi_R} \times \boldsymbol{r} \end{aligned} \tag{4.22}$$

Now we try to find unitary operator and generators for rotation.

$$\begin{aligned} \hat{U}_R(\psi.\hat{n})\Psi(\boldsymbol{r}) &= \Psi'(\boldsymbol{r}) \\ \hat{U}_R(\psi.\hat{n})\Psi(\boldsymbol{r}) &= \Psi(\boldsymbol{r} - \boldsymbol{\delta r}) \end{aligned} \tag{4.23}$$

Using Taylor series expansion,

$$\hat{U}_R(\psi.\hat{n})\Psi(\boldsymbol{r}) = \Psi(\boldsymbol{r}) - \boldsymbol{\delta r}.\nabla\Psi(\boldsymbol{r}) \tag{4.24}$$

$$\hat{U}_R(\psi.\hat{n})\Psi(\boldsymbol{r}) = \Psi(\boldsymbol{r}) - (\boldsymbol{\delta\psi} \times \boldsymbol{r}).\nabla\Psi(\boldsymbol{r}) \tag{4.25}$$

We have

$$\nabla\Psi(\boldsymbol{r}) = \frac{\iota}{\hbar}\boldsymbol{\hat{p}}\Psi(\boldsymbol{r}) \tag{4.26}$$

where

$$\hat{\boldsymbol{p}} = \frac{\hbar}{\iota}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]$$

$$\hat{U}_R(\psi.\hat{n})\Psi(\boldsymbol{r}) = \left[1 - (\frac{\iota}{\hbar}\boldsymbol{\delta\psi} \times \boldsymbol{r}).\hat{\boldsymbol{p}}\right]\Psi(\boldsymbol{r}) \tag{4.27}$$

$$\hat{U}_R(\psi.\hat{n})\Psi(\boldsymbol{r}) = \left[1 - \frac{\iota}{\hbar}\boldsymbol{\delta\psi}.(\boldsymbol{r} \times \hat{\boldsymbol{p}})\right]\Psi(\boldsymbol{r}) \tag{4.28}$$

In the above equation the term$\boldsymbol{r} \times \boldsymbol{p}$ can be identified as orbital angular momentum.Comparing with 4.10 we can conclude that

$$\hat{L}_x, \hat{L}_y, \hat{L}_z \longrightarrow generators$$

$$\delta\psi_x, \delta\psi_y, \delta\psi_z \longrightarrow parameters$$

for rotations in 3D space.

5 Flavour symmetry

In the early days of nuclear physics, it was realised that the proton and neutron have very similar masses and that the nuclear force is approximately charge independent.To reflect this observed symmetry of the nuclear force, it was proposed that the neutron and proton could be considered as two states of a single entity, the nucleon, analogous to the spin-up and spin-down states of a spin-half particle.

$$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This led to the introduction of the idea of isospin, where the proton and neutron form an isospin doublet with total isospin I = 1/2 and third component of isospin $I_3 = \pm 1/2$. The charge independence of the strong nuclear force is then expressed in terms of invariance under unitary transformations in this isospin space.

The idea of proton/neutron isospin symmetry can be extended to the quarks. Since the QCD interaction treats all quark flavours equally, the strong interaction possesses a flavour symmetry analogous to isospin symmetry of the nuclear force.
The above idea can be developed mathematically by writing the up and down quarks as states in an abstract flavour space.

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If the up- and down-quarks were indistinguishable, the flavour independence of the QCD interaction could be expressed as an invariance under a general unitary transformation in this abstract space

$$\begin{bmatrix} u' \\ d' \end{bmatrix} = \hat{U} \begin{bmatrix} u \\ d \end{bmatrix} \tag{5.1}$$

The operator $\hat{U}$ must satisfiy $\hat{U}\hat{U}^{\dagger}$ belongs to the U(2) group.This condition leads to 4 independent parameters and hence 4 generators corresponding to unitary operator $\hat{U}$ 3 of which belong to SU(2) subgroup. A suitable choice for three Hermitian traceless generators of the ud flavour symmetry are the Pauli spin-matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -\iota \\ \iota & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The ud flavour symmetry corresponds to invariance under SU(2) transformations leading to three conserved observable quantities defined by the eigenvalues of Pauli spin-matrices. The algebra of the ud flavour symmetry is therefore identical to that of spin for a spin-half particle. In analogy with the quantum-mechanical treatment of spin-half particles, isospin $\hat{\boldsymbol{T}}$ is defined in terms of the Pauli spin-matrices

$$\hat{\boldsymbol{T}} = \frac{1}{2}\boldsymbol{\sigma} \tag{5.2}$$

Any finite transformation in the up–down quark flavour space can be written in terms of a unitary transformation

$$\hat{U} = e^{\iota \boldsymbol{\alpha}.\hat{\boldsymbol{T}}} \tag{5.3}$$

Hence, the general flavour transformation is a “rotation” in flavour space

5.1 Isospin algebra

The three generators of the group, which correspond to physical observables, satisfy the algebra

$$[\hat{T}_1, \hat{T}_2] = \iota\hat{T}_3 \tag{5.4}$$

$$[\hat{T}_2, \hat{T}_3] = \iota\hat{T}_1 \tag{5.5}$$

$$[\hat{T}_3, \hat{T}_1] = \iota\hat{T}_2 \tag{5.6}$$

This is exactly the same set of commutators as found for the quantum mechanical treatment of angular momentum.The total isospin operator,

$$\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2 \tag{5.7}$$

which commutes with each of the generators, is Hermitian and therefore also corresponds to an observable quantity. Because the three operators $\hat{T}_1, \hat{T}_2$ and

$\hat{T}_3$ do not commute with each other,the corresponding observables cannot be known simultaneously.Hence, isospin states can be labelled in terms of the total isospin I and the third component of isospin I_3. These isospin states $\psi(I, I_3)$ are the mathematical analogues of the angular momentum states $\langle l, m|$and have the properties

$$\hat{T}^2\phi(I, I_3) = I(I+1)\phi(I, I_3) \quad (5.8)$$

$$\hat{T}_3\phi(I, I_3) = I_3\phi(I, I_3) \quad (5.9)$$

In terms of isospin, the up-quark and down-quark are represented by

$$u = \begin{bmatrix}1\\0\end{bmatrix} = \phi(\frac{1}{2}, +\frac{1}{2}) \quad (5.10)$$

$$d = \begin{bmatrix}0\\1\end{bmatrix} = \phi(\frac{1}{2}, -\frac{1}{2}) \quad (5.11)$$

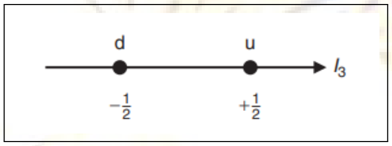

Figure 5.1.1: The isospin one-half multiplet consisting of an up-quark and a down-quark.

The isospin ladder operators are defined as

$$\hat{T}_- \equiv \hat{T}_1 - \iota\hat{T}_2 \quad (5.12)$$

$$\hat{T}_+ \equiv \hat{T}_1 + \iota\hat{T}_2 \quad (5.13)$$

The action the ladder operators on a particular isospin state are

$$\hat{T}_+\phi(I, I_3) = \sqrt{I(I+1) - I_3(I_3+1)}\phi(I, I_3+1) \quad (5.14)$$

$$\hat{T}_-\phi(I, I_3) = \sqrt{I(I+1) - I_3(I_3-1)}\phi(I, I_3-1) \quad (5.15)$$

In case of extreme states with $I_3 = \pm I$

$$\hat{T}_-\phi(I, -I) = 0 \quad (5.16)$$

$$\hat{T}_+\phi(I, +I) = 0 \quad (5.17)$$

So we can conclude that the effects of isospin ladder operators on the u- and d- quarks are

$$\hat{T}_+ u = 0, \hat{T}_+ d = u, \hat{T}_- u = d \text{ and } \hat{T}_- d = 0$$

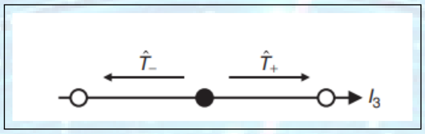

Figure 5.1.2: The isospin ladder operators step along the states in I_3 within an isospin multiplet

5.2 Combining quarks into baryons

5.2.1 Combination of two quarks

As every symmetry is associated with a conserved observable quantity(measured by the corresponding generators),in the case of flavour symmetry in strong interactions also we have I and I_3 as the conserved observables.Because I_3 and I are conserved in strong interactions, the concept of isospin is useful in describing low energy hadron interactions. Here we'll use the concept of isospin to construct the flavour wavefunctions of baryons (qqq) and mesons ($q\bar{q}$).
when combining a system of two quarks the third component of isospin is added as a scalar and the total isospin is added as the magnitude of a vector.r. If two isospin states $\phi(I^a, I_3^a)$ and $\phi(I^b, I_3^b)$ are combined, the resulting isospin state (I, I_3) has

$$I_3 = I_3^a + I_3^b \tag{5.18}$$

$$|I^a - I^b| \leq I \leq |I^a + I^b| \tag{5.19}$$

The I_3 assignments of the four possible combinations of two light quarks are shown in figure 5.2.1.The isospin assignments for the extreme states immediately can be identified as

$$uu = \phi(1, +1) \text{ and } dd = \phi(1, -1)$$

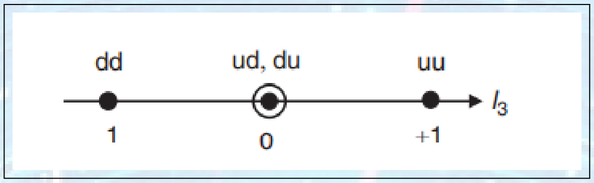

Figure 5.2.1: The I_3 assignments for the four possible combinations of two up- or down-quarks

The quark combinations ud and du, which both have $I_3 = 0$, are not eigenstates of total isospin. The appropriate linear combination corresponding to the $I = 1$ state can be identified using isospin ladder operators,

$$\hat{T}_-\phi(1,+1) = \sqrt{2}\phi(1,0) = \hat{T}_-(uu) = ud + du \tag{5.20}$$

$$\implies \phi(1,0) = \frac{1}{\sqrt{2}}(ud + du) \tag{5.21}$$

The $\phi(0,0)$ state can be identified as the linear combination of ud and du that is orthogonal to $\phi(1,0)$, from which

$$\phi(0,0) = \frac{1}{\sqrt{2}}(ud - du) \tag{5.22}$$

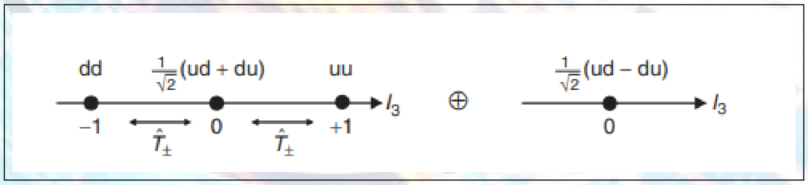

Figure 5.2.2: The isospin assignments for the combinations of two quarks

The four possible combinations of two isospin doublets therefore decomposes into a triplet of isospin-1 states and a singlet isospin-0 state, as shown in Figure 5.2.1. This decomposition can be written as $2 \otimes 2 = 3 \oplus 1$.Note that isospin-0 and isospin-1 states are physically different; the isospin-1 triplet is symmetric under interchange of the two quarks, whereas the isospin singlet is antisymmetric.

5.2.2 Combination of three quarks

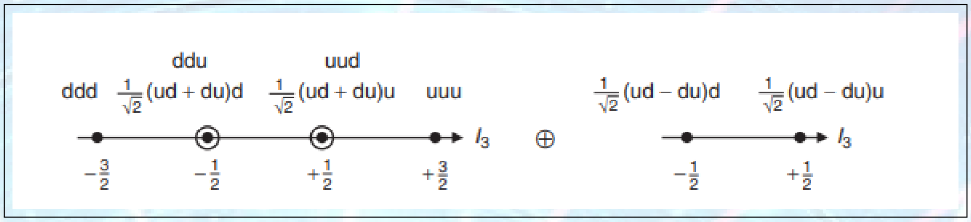

Figure 5.2.3: The I_3 assignments for three quark system built from qq triplet and singlet states

The isospin states formed from three quarks can be obtained by adding an up or down-quark to the qq isospin singlet and triplet states of Figure 5.2.1. Since I_3 adds as a scalar, the I_3 assignments of the possible combinations are those shown in Figure 5.2.2.
The two states built from the $I = 0$ singlet will have total isospin $I = 1/2$, whereas those constructed from the $I = 1$ triplet can have either $I = 1/2$ or $I = 3/2$. Of the six combinations formed from the triplet, the extreme ddd and uuu states with $I_3 = -3/2$ and $I_3 = +3/2$ uniquely can be identified as being part of isospin $I = 3/2$ multiplet. The other two $I = 3/2$ states can be identified using the ladder operators.
For example, the $\phi\left(\frac{3}{2}\right)$ state , which is a linear combination of the ddu and $\frac{1}{\sqrt{2}}$(ud + du)d states, can be obtained from the action of $\hat{T}_+$ as follows

$$\hat{T}_+\phi\left(\frac{3}{2},-\frac{3}{2}\right) = \sqrt{3}\phi\left(\frac{3}{2},-\frac{1}{2}\right) = \hat{T}_+(ddd) \tag{5.23}$$

$$= udd + dud + ddu \tag{5.24}$$

$$\Longrightarrow \boxed{\phi\left(\frac{3}{2},-\frac{1}{2}\right) = \frac{1}{\sqrt{3}}(uud + dud + ddu)} \tag{5.25}$$

Similarly by operation of ladder operator $\hat{T}_-$ on $\hat{T}_-\phi\left(\frac{3}{2},\frac{3}{2}\right) = uuu$ gives us the state $\phi\left(\frac{3}{2},\frac{1}{2}\right)$.

Finally the four isospin $\frac{3}{2}$ states constructed from the qqq triplet are

$$\phi\left(\frac{3}{2},-\frac{3}{2}\right) = ddd \tag{5.26}$$

$$\phi\left(\frac{3}{2},-\frac{1}{2}\right) = \frac{1}{\sqrt{3}}(udd + dud + ddu) \tag{5.27}$$

$$\phi\left(\frac{3}{2},+\frac{1}{2}\right) = \frac{1}{\sqrt{3}}(uud + udu + duu) \tag{5.28}$$

$$\phi\left(\frac{3}{2},+\frac{3}{2}\right) = uuu \tag{5.29}$$

The two states obtained from the qqq triplet with total isospin $I = 1/2$ are orthogonal to the $I_3 = \pm 1/2$ states of 5.27 and 5.28 respectively.Hence, the $\phi\left(\frac{1}{2},-\frac{1}{2}\right)$ state can be identified as the linear combination of ddu and $\frac{1}{\sqrt{2}}$ (ud + du)d that is orthogonal to the $\phi\left(\frac{3}{2},-\frac{1}{2}\right)$ state of 5.27, giving

$$\phi_S\left(\frac{1}{2},-\frac{1}{2}\right) = -\frac{1}{\sqrt{6}}(2ddu - udd - dud) \tag{5.30}$$

Similarly,

$$\phi_S\left(\frac{1}{2},+\frac{1}{2}\right) = \frac{1}{\sqrt{6}}(2uud - udu - duu) \tag{5.31}$$

The two states constructed from the qq isospin singlet of 5.22 are

$$\phi_A\left(\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{\sqrt{2}}(udd - dud) \tag{5.32}$$

$$\phi_A\left(\frac{1}{2},+\frac{1}{2}\right) = \frac{1}{\sqrt{2}}(udu - duu) \tag{5.33}$$

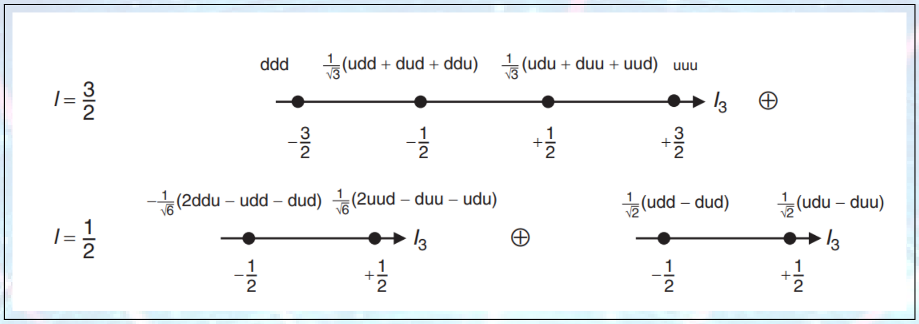

Figure 5.2.4: The three quark states(qqq) in SU(2) flavoug symmetry

Hence, the eight combinations of three up- and down-quarks, uuu, uud, udu, udd, duu, dud, ddu and ddd, have been grouped into an isospin- $\frac{3}{2}$ quadruplet and two isospin- $\frac{1}{2}$ doublets, as shown in Figure 5.2.2.
In terms of the SU(2) group structure this can be expressed as

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

where $2 \otimes 2 \otimes 2$ represents the combinations of three quarks represented as isospin doublets.

The different isospin multiplets have different exchange symmetries. The flavour states in the isospin -$\frac{3}{2}$ quadruplet, $5.26 - 5.29$,are symetric under the interchange of any two quarks.The isospin- $\frac{1}{2}$ doublets are referred to as mixed symmetry states to reflect the symmetry under the interchange of the first two quarks, but lack of overall exchange symmetry.

5.3 Combining quarks into mesons

A mesons is a bound state of a quark and an antiquark($q\bar{q}$).We are already familiar with isospin represntation of quarks so let's have a look at isospin representations for antiquarks.

5.3.1 Isospin represntations of antiquarks

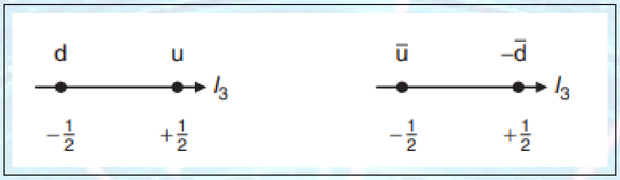

Figure 5.3.1: The isospin representation of d and u quarks and $\bar{d}$ and $\bar{u}$ quarks

In the above description of SU(2) flavour symmetry, the up- and down-quarks were placed in an isospin doublet

$$q = \begin{bmatrix} u \\ d \end{bmatrix}$$

A general SU(2) transformation of the quark doublet, $q \longrightarrow q' = Uq$,(where U $\in$ SU(2) group) can be written

$$\begin{bmatrix} u \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} u' \\ d' \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \cdot \begin{bmatrix} u \\ d \end{bmatrix} \tag{5.34}$$

Hence taking the complex conjugate of 5.34 gives the transformation properties of the flavour part of the antiquark wavefunctions

$$\begin{bmatrix} \bar{u'} \\ \bar{d'} \end{bmatrix} = U^* \begin{bmatrix} \bar{u} \\ \bar{d} \end{bmatrix} = \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} \bar{u} \\ \bar{d} \end{bmatrix} \tag{5.35}$$

In SU(2) it is possible to place the antiquarks in a doublet that transforms in the same way as the quarks, $\bar{q} \longrightarrow \bar{q'} = U\bar{q}$. If the antiquark doublet is written as

$$\bar{q} \equiv \begin{bmatrix} -\bar{d} \\ \bar{u} \end{bmatrix} = S \begin{bmatrix} \bar{u} \\ \bar{d} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{d} \end{bmatrix} \tag{5.36}$$

Since

$$\begin{bmatrix} \bar{u} \\ \bar{d} \end{bmatrix} = S^{-1}\bar{q} \text{ and } \begin{bmatrix} \bar{u'} \\ \bar{d'} \end{bmatrix} = S^{-1}\bar{q'},$$

Rewriting 5.35

$$S^{-1}\bar{q'} = U^*S^{-1}\bar{q}$$
$$\bar{q'} = SU^*S^{-1}\bar{q}$$
$$\boxed{\bar{q'} = U\bar{q}} \tag{5.37}$$

Hence, by placing the antiquarks in an SU(2) doublet defined by

$$\boxed{\bar{q} \equiv \begin{bmatrix} -\bar{d} \\ \bar{u} \end{bmatrix}}$$

the antiquarks transform in exactly the same manner as the quarks.It also ensures that quarks and antiquarks behave in the same way under SU(2) flavour transformations and that physical predictions are invariant under the simultaneous transformations of the form $u \leftrightarrow$ d and $\bar{u} \leftrightarrow \bar{d}$.
The effect of the isospin ladder operators on the antiquark doublet can be seen to be

$$T_+\bar{u} = -\bar{d}, T_+\bar{d} = 0, T_-\bar{u} = 0 \text{ and } T_-\bar{d} = -\bar{u}$$

5.3.2 Meson States

In terms of isospin, the four possible states formed from up- and down-quarks/antiquarks can be expressed as the combination of an SU(2) quark doublet and an SU(2) antiquark doublet. Using the isospin assignments of Figure 5.3.1, the du state immediately can be identified as the $q\bar{q}$ isospin state, $\phi(1,-1)$. Application of isospin ladder operator $\hat{T}_+$ yields other states as well,

$$\phi(1,-1) = d\bar{u} \tag{5.38}$$
$$\phi(1,0) = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \tag{5.39}$$
$$\phi(1,+1) = -u\bar{d} \tag{5.40}$$

The isospin singlet,which must be orthogonal to the $\phi(1,0)$ state,is therefore

$$\phi(0,0) = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \tag{5.41}$$

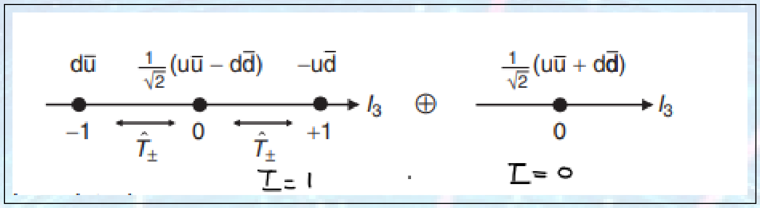

Figure 5.3.2: The $q\bar{q}$ isospin triplet and singlet states

This decomposition into an isospin triplet and an isospin singlet is expressed as $2 \otimes bar2 = 3 \oplus 1$, where the 2 is the isospin representation of the quark doublet and the $\bar{2}$ is the isospin representation of doublet.

6 Summary

In this report we started with a basic introduction to different types of particles ,their properties and the ways they are classified.Then we looked at dynamics of these particles and studied different conservation laws which can be used to predict the outcome of a particular nucler reaction.In the subsequent sections we introduced Lie groups and the concept of generators. Later we analysed Flavour Symetry and introduced isospin as the corresponding conserved quantity to construct bound states of combination of three quarks(Baryon) and a quark and an antiquark(Meson).

We've seen how to construct bound states by taking up and down quarks and working with 2×2 unitary matrices (SU(2) flavour symmetry).We can extend this further by taking into account strange quarks as well and thus invoking SU(3) flavour symmetry . This can ultimately be used to generate different quark models viz; Baryon octets and decuplets and Meson nonets.

7 References

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