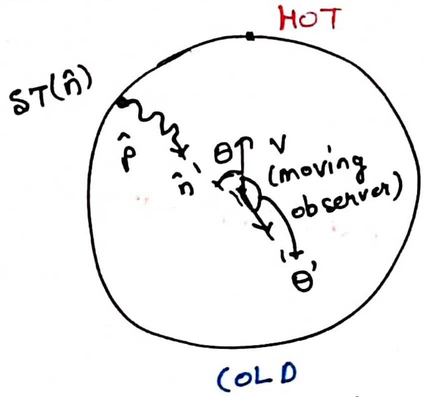
CMB Notes → Baumann

7.1 Anisotropies in the First Light

→ Largest anisotropy ⟶ Temp. dipole of mag. 3.36 mK, due to motion of the solar system w.r.t. rest frame of CMB



Moving frame → S' (moving with velocity $\vec{v}$)

$$E = \gamma(E' + \vec{v}\cdot\vec{p}')$$

For a photon $E' = p'c$

also $E = k_B T$

$E' = k_B T'$

$$T = \gamma(T')\left(1 + \frac{v}{c}\cos\theta'\right)$$

$$T' = \frac{T}{\gamma\left(1 + \frac{v}{c}\cos\theta'\right)} = T\left(1 + \frac{v}{c}\cos\theta + O\left(\frac{v^2}{c^2}\right)\right)$$

7.1 Angular Power Spectrum

$$T(\hat{n}) = \bar{T}_0\left(1 + \Theta(\hat{n})\right)$$

$C(\theta) = \langle \Theta(\hat{n})\Theta(\hat{n}')\rangle$ ⟶ Two point correlation fn
↳ ∠ avged over whole sky

$$\Theta(\hat{n}) = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$$ ↳ multipole moments

$\left\{ Y^*_{\ell m} = (-1)^m Y_{\ell,-m},\quad a^*_{\ell m} = (-1)^m a_{\ell,-m} \right\}$

$\ell = 0$ is monopole
$\ell = 1$ dipole → already sub.

$$\langle a_{\ell m}, a^*_{\ell' m'}\rangle = C_\ell \delta_{\ell\ell'}\delta_{mm'}$$ ①

↳ C_ℓ: $\boxed{\text{Angular Power Spectrum}}$

$$C(\theta) = \langle \Theta(\hat{n})\,\Theta(\hat{n}')\rangle$$

$$= \sum_{\ell m}\sum_{\ell' m'} \langle a_{\ell m} a^*_{\ell' m'}\rangle Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}')$$

$$= \sum_\ell C_\ell \sum_m Y_{\ell m}(\hat{n}) Y^*_{\ell m}(\hat{n}')$$

$$\boxed{C(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta)} \Leftrightarrow \boxed{C_\ell = 2\pi\int_{-1}^{1} d(\cos\theta)\, C(\theta) P_\ell(\cos\theta)}$$

– (7.5)

Variance of temp. anisotropy field.

↳ $C(0) = \sum_{\ell} \frac{2\ell+1}{4\pi} C_\ell \approx \int d \ln \ell \, \frac{\ell(\ell+1) C_\ell}{2\pi}$

Power per logarithmic interval in ℓ is

$$\Delta_T^2 = \sigma^2 \bar{T}_0^2$$

$$\boxed{\Delta_T^2 = \frac{\ell(\ell+1) C_\ell \bar{T}_0^2}{2\pi}} = D_\ell \text{ (Planck Papers)}$$

If fluctuations scale invariant $\Rightarrow \Delta_T^2$ independent of ℓ.

Cosmic Variance D

→ Even though we measure $\Theta(\hat{n})$ precisely, can't measure C_ℓ

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2$$

$$\Rightarrow \frac{\Delta C_\ell}{C_\ell} = \frac{\sqrt{\langle (C_\ell - \hat{C}_\ell)^2 \rangle}}{C_\ell} = \sqrt{\frac{2}{2\ell+1}}$$ ↗ var ↑ as $\ell \downarrow$

↳ (less no. of samples)

7.2 Photons in a Clumpy Universe

7.2.1 Grav. Redshift

$$\frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\nu\rho} p^\nu p^\rho \qquad \left\{ p^\mu = \frac{dx^\mu}{d\lambda} \right\}$$

Newtonian gauge: $ds^2 = a^2(\eta) \left[-(1+2\Psi) d\eta^2 + (1-2\Phi) dx^2 \right]$

$$\frac{dp^\mu}{d\lambda} = \frac{d\eta}{d\lambda} \frac{dp^\mu}{d\eta} = p^0 \frac{dp^\mu}{d\eta}$$

For evolution of photon energy, we consider $\mu = 0$

$$\frac{dp^0}{d\eta} = -\Gamma^0_{\nu\rho} \frac{p^\nu p^\rho}{p^0} = -\Gamma^0_{00} p^0 - 2\Gamma^0_{0i} p^i - \Gamma^0_{ij} \frac{p^i p^j}{p^0}$$

$$= -(\mathcal{H} + \Psi') p^0 - 2 \partial_i \Phi p^i - [\mathcal{H} - \Phi' - 2\mathcal{H}(\Phi + \Psi)] \delta_{ij} \frac{p^i p^j}{p^0}$$

↳ $\left(\frac{d\Psi}{d\eta}\right)$ $\left(\mathcal{H} = \frac{a'}{a}\right)$ — (7.15)

Note $p^\mu = (p^0, p^i) \rightarrow$ Coordinate frame

$p^{\hat\mu} = (E, p^{\hat i}) \rightarrow$ Observer's local inertial frame

$$\eta_{\hat\mu\hat\nu} p^{\hat\mu} p^{\hat\nu} = g_{\mu\nu} p^\mu p^\nu$$

$$-E^2 + \delta_{ij} p^{\hat i} p^{\hat j} = g_{00}(p^0)^2 + g_{ij} p^i p^j$$

$$\Rightarrow \quad E = \sqrt{-g_{00}}\, p^0$$

$$p^2 = g_{ij} p^i p^j = \delta_{ij} p^{\hat i} p^{\hat j}$$

$\cancel{p^0}$

$$p^0 = \frac{E}{\sqrt{-g_{00}}} = \frac{E}{\sqrt{a^2(1+2\Psi)}} = \frac{E}{a}(1-\Psi)$$

$$p^i = \frac{E}{\sqrt{g_{ii}}}\hat p^i = \frac{E}{a}(1+\Phi)\,\hat p^i$$

Unit vector

Subs in (7.15) & Consider I order

We get

$$\boxed{\frac{1}{E}\frac{dE}{d\eta} = -\mathcal{H} + \Phi' - \hat p^i \partial_i \Psi}$$

Redshifting of photon energy (due to expansion of universe)

Can be viewed as local perturbation to scale factor $\tilde a(\eta, \vec x) = a(\eta)(1-\Phi(\vec x))$, $E \propto \tilde a^{-1}$

Grav. Redshift as photon travels out of a pot. well.

Note : $\hat p^i \cancel{\neq} \frac{dx^i}{d\eta} = \frac{dx^i}{d\lambda}\frac{d\lambda}{d\eta} = \frac{p^i}{p^0} = \frac{(1+\Phi)\hat p^i}{(1-\Psi)} = \hat p^i$

$$\Rightarrow \quad \hat p^i \partial_i \Psi = \frac{dx^i}{d\eta}\frac{\partial \Psi}{\partial x^i} = \frac{d\Psi}{d\eta} - \frac{\partial \Psi}{\partial \eta} = \frac{d\Psi}{d\eta} - \Psi'$$

$$\boxed{\frac{d(\ln aE)}{d\eta} = -\frac{d\Psi}{d\eta} + \Phi' + \Psi'} \quad - \quad (7.23)$$

7.2.2 Line-of-Sight Solution

→ Assuming instantaneous photon decoupling (at some $\eta = \eta_*$) integrate (7.23) to relate photon energy at decou-pling & today.

$$\ln(aE)_0 = \ln(aE)_* - (\Phi_0 - \Psi_*) + \int_{\eta_*}^{\eta_0} d\eta\,(\Phi' + \Psi')$$

~~ln~~ $aE \propto a(\bar T + \delta T)$

$$\ln(a\bar T)_0 + \ln\left(1+\frac{\delta T}{\bar T}\right)_0 = \ln(a\bar T)_* + \ln\left(1+\frac{\delta T}{\bar T}\right)_* - (\Psi_0 - \Psi_*) + \int_{\eta_*}^{\eta_0} d\eta\,(\Phi' + \Psi')$$

$(a\bar T)_0 = (a\bar T)_*$

$$\boxed{\left(\frac{\delta T}{\bar T}\right)_0 = \left(\frac{\delta \bar T}{\bar T}\right)_* + \Psi_* + \int_{\eta_*}^{\eta_0} d\eta\,(\Phi' + \Psi')}$$

$\Psi_0 \to$ dropped $\to$ only contributes to the monopole term (??)
↳ at ($k=0$) (no dir^n dep.)

7.2.3 Fluctuations at last scattering

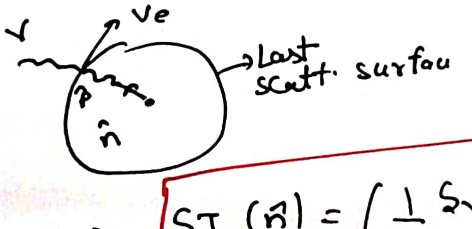
The $\left(\frac{\delta T}{T}\right)_*$ term has two main contributions

→ Photon density at recomb^n $\to \rho_\gamma \propto T^4 \Rightarrow \frac{\delta T}{T} \propto \frac{\delta_\gamma}{4}$

$\left(\delta_\gamma = \frac{\delta\rho_\gamma}{\rho_\gamma}\right)$

→ Due to bulk velocity of $e^\ominus$ at recomb^n $\to \frac{\delta T}{T} \propto \hat p \cdot \vec v_e = -\hat n \cdot \vec v_B$

(baryon - $e^\ominus$ tightly coupled)



$$\Rightarrow \boxed{\frac{\delta T}{T}(\hat n) = \underbrace{\left(\frac{1}{4}\delta_\gamma + \Phi\right)_*}_{\text{Sachs-Wolfe (SW)}} \underbrace{- (\hat n \cdot \vec v_b)_*}_{\text{Doppler}} + \underbrace{\int_{\eta_0}^{\eta_0} d\eta\,(\Phi' + \Phi')}_{\text{Integrated SW}}} \quad -(7.20)$$

7.3 Anisotropies from Inhomogenities

7.3.1 Spatial to Angular projection

→ Inhomogenities in Primordial Plasma ⟶ Correlations b/w Temps in diff. dirns.

→ Switch to Fourier Space (Fig 7.8 helpful)

→ Ignore ISW contri. for now ⇒ fluctuations in the dirn $\hat{n}$ is directly related to the fluctuation in the perturbations at $x_* = \chi_* \hat{n}$

→ $\vec{v}_b = i\hat{k} v_b$ (92)

→ FT of (7.28) gives (7.29)

$$\Theta(\hat{n}) = \frac{\delta T}{T}(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\chi_*\hat{n})} \left[F(\eta_*,\vec{k}) - i(\hat{k}\cdot\hat{n})\, G(\eta_*,\vec{k})\right]$$

$F = \frac{1}{4}\delta_\gamma + \Phi$; $G = v_b$ → Factor out $\mathcal{R}_i(\vec{k}) = \mathcal{R}(0,\vec{k})$ initial curvature pert.

$$\Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\chi_*\hat{n})} \left[F_*(k) - i(\hat{k}\cdot\hat{n})\, G_*(k)\right] \mathcal{R}_i(\vec{k})$$

$$\boxed{F_*(k)} = \frac{F(\eta_*,\vec{k})}{\mathcal{R}_i(\vec{k})} \qquad \boxed{G_*(k)} = \frac{G(\eta_*,\vec{k})}{\mathcal{R}_i(\vec{k})}$$

SW T.F. Doppler T.F.

(also $\chi_* \hat{n} = \vec{x}_*$) Transfer Functions

Using $e^{i(\vec{k})\cdot(\chi_*\hat{n})} = \sum_\ell i^\ell (2\ell+1)\, j_\ell(k\chi_*)\, P_\ell(\hat{k}\cdot\hat{n})$

↳ Spherical Bessel function

for the doppler T.F. $\hat{k}\cdot\hat{n}$ can be written as

$$i(\hat{k}\cdot\hat{n})\, e^{i\vec{k}\cdot(\chi_*\hat{n})} = \frac{d}{d(k\chi_*)} e^{i\vec{k}\cdot(\chi_*\hat{n})} = \sum_\ell i^\ell (2\ell+1)\, j_\ell'(k\chi_*)\, P_\ell(\hat{k}\cdot\hat{n})$$

$$\Rightarrow \Theta(\hat{n}) = \sum_\ell i^\ell (2\ell+1) \int \frac{d^3k}{(2\pi)^3} \Theta_\ell(k)\, \mathcal{R}_i(\vec{k})\, P_\ell(\hat{k}\cdot\hat{n})$$

where $\Theta_\ell(k) = F_*(k)\, j_\ell(k\chi_*) - G_*(k)\, j_\ell'(k\chi_*)$

Plug this into the exp. of two point fn., we get

$$\langle\Theta(\hat{n})\,\Theta(\hat{n}')\rangle = \cancel{\sum_\ell} \Big\langle \sum_\ell i^\ell (2\ell+1) \int \frac{d^3k}{(2\pi)^3} \Theta_\ell(k)\, \mathcal{R}_i(\vec{k})\, P_\ell(\hat{k}\cdot\hat{n}) \sum_{\ell'} i^{\ell'} (2\ell'+1) \int \frac{d^3k'}{(2\pi)^3} \Theta_{\ell'}(k')\, \mathcal{R}_i(\vec{k}')\, P_{\ell'}(\hat{k}'\cdot\hat{n}') \Big\rangle$$

$$= \sum_{l}\sum_{l'} i^{l+l'} (2l+1)(2l'+1) \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3}\, \Theta_l(k)\, \Theta_{l'}(k')\, P_l(\hat{k}\cdot\hat{n})\, P_{l'}(\hat{k}'\cdot\hat{n}')\, \frac{2\pi^2}{k^3} \Delta^2_{\mathcal{R}}(k)\, (2\pi)^3\, \delta_D(\vec{k}+\vec{k}')$$

$\hookrightarrow$ (Using $\langle \mathcal{R}_i(\vec{k})\, \mathcal{R}_i(\vec{k}')\rangle = \frac{2\pi^2}{k^3} \Delta^2_{\mathcal{R}}(k)\, (2\pi)^3\, \delta_D(\vec{k}+\vec{k}')$)

$$= \sum_{l}\sum_{l'} i^{l+l'} (2l+1)(2l'+1) \int \frac{d^3\vec{k}}{4\pi^2}\, \Theta_l(k)\, \Theta_{l'}(k)\, P_l(\hat{k}\cdot\hat{n})\, P_{l'}(-\hat{k}\cdot\hat{n}')\, \Delta^2_{\mathcal{R}}(k)/k^3$$

$\hookrightarrow$ ($\vec{k}' = -\vec{k} \Rightarrow \hat{k}' = -\hat{k}$, $k' = k$)

Using $\int d^2\hat{k}\, P_l(\hat{k}\cdot\hat{n})\, P_{l'}(\hat{k}\cdot\hat{n}') = \frac{4\pi}{2l+1} P_l(\hat{n}\cdot\hat{n}')\, \delta_{ll'}$

& $P_{l'}(-\hat{k}\cdot\hat{n}') = (-1)^{l'} P_{l'}(\hat{k}\cdot\hat{n}')$

$$= \sum_{l}\sum_{l'} i^{l+l'} (2l+1)(2l'+1)(-1)^{l'} \int \frac{dk}{4\pi^2 k}\, \Theta_l(k)\, \Theta_{l'}(k)\, \Delta^2_{\mathcal{R}}(k)\, \frac{4\pi}{2l+1} P_l(\hat{n}\cdot\hat{n}')\, \delta_{ll'}$$

$$= \sum_{l} i^{2l} (-1)^l (2l+1) \int \frac{dk}{4\pi^2 k} \cdot 4\pi\, \Theta_l^2(k)\, \Delta^2_{\mathcal{R}}(k)\, P_l(\hat{n}\cdot\hat{n}')$$

$$= \sum_{l} \frac{(2l+1)}{4\pi} \left[4\pi \int d(\ln k)\, \Theta_l^2(k)\, \Delta^2_{\mathcal{R}}(k) \right] P_l(\hat{n}\cdot\hat{n}')$$

Comparing with (7.8) we get

$$\boxed{C_l = 4\pi \int d(\ln k)\, \underbrace{\Theta_l^2(k)}_{\text{T.F.}}\, \underbrace{\Delta^2_{\mathcal{R}}(k)}_{\text{Power Spectrum of Primordial Curvature pert.}}} \qquad -(7.39)$$

Including ISW term we get

$$\Theta_l(k) = F_*(k)\, j_l(kK_*) - G_*(k)\, j_l'(kK_*) + \int_{\eta_*}^{\eta_0} d\eta\, (\Phi' + \Psi')\, j_l(kK)$$

where $K(\eta) = \eta_0 - \eta$

7.3.2 Large Scales : Sachs-Wolfe Effect

→ Low poles of CMB ($\ell < 100$) are created by super-horizon fluctuations at recombination. (dominated by Sachs-Wolfe term)

→ Super-horizon limit $\Rightarrow \boxed{\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_c = \frac{4}{3}\delta_b = -2\Phi_i}$ $\left(-2\Phi \simeq \delta \simeq \delta_m = \frac{3}{4}\delta_\gamma\right)$

(Neglecting anisotropic stress $\Rightarrow \Phi_i = \Psi_i$) initial pot.

$\Rightarrow \Theta(\hat{n}) \approx \left(\frac{1}{4}\delta_\gamma + \Psi\right)_* = \frac{1}{3}\Psi_* = \frac{1}{5}\mathcal{R}_* \rightarrow$ (Matter dominated era $\rightarrow \mathcal{R} \rightarrow \frac{5}{3}\Phi$)

↳ observed CMB Temp. fluctuations on large scale

Note : (i) There has been no evolution on large scales ($\because$ they've not yet entered horizon) thus this limit directly probes the initial condⁿ.

(ii) Grav. Redshift Ψ_* is greater than $\frac{1}{4}\delta_\gamma$ contribution.

Hence for an overdensity ($\delta_{\gamma *} > 0$) ($\Psi_* < 0$) $\Rightarrow$ ~~Cold~~ $\Rightarrow$ Cold spot

Underdensity ($\delta_{\gamma *} < 0$) ($\Psi_* > 0$) $\Rightarrow$ Hot spot

Comparing with 7.29, we have $F = \frac{1}{5}\mathcal{R}_*$ & $F_* = \frac{1}{5}\frac{\mathcal{R}_*(k)}{\mathcal{R}_i(k)}$, there is no evolution

For superhorizon scales

$\Rightarrow F_* = \frac{1}{5}$ $\Rightarrow \boxed{\Theta_\ell^{SW}(k) = \frac{1}{5} j_\ell(k\chi_*)}$

Power spectrum $C_\ell = \frac{4\pi}{25}\int d(\ln k)\; \Delta^2_{\mathcal{R}}(k)\, j_\ell^2(k\chi_*)$

Considering a primordial spectrum of power law form $\Delta^2_{\mathcal{R}}(k) = A_s\left(\frac{k}{k_0}\right)^{n_s - 1}$, the integral can be evaluated analytically.

$$C_\ell^{SW} = \frac{4\pi}{25} A_s (k_0\chi_*)^{1-n_s}\, 2^{n_s-4} \underbrace{\frac{\Gamma(3-n_s)}{\Gamma^2(4-\frac{n_s}{2})}}_{T_1} \underbrace{\frac{\Gamma\left(\ell + \frac{n_s-1}{2}\right)}{\Gamma\left(\ell + 2 - \frac{n_s-1}{2}\right)}}_{T_2}$$

$\Rightarrow$ For $n_s = 1$ $T_1 = \frac{\Gamma(2)}{\Gamma^2(3/2)} = \frac{4}{\pi}$, $T_2 = \frac{\Gamma(\ell)}{\Gamma(\ell+2)} = \frac{1}{\ell(\ell+1)}$

& also scale dep. from $(k_0\chi_*)^{1-n_s}$ disappear.

Thus we get $\Rightarrow \Delta_T^2 = \frac{\ell(\ell+1)}{2\pi} C_\ell \bar{T}_0^2 = \frac{A_s}{25}\bar{T}_0^2 \rightarrow$ (Constant)

↳ (Can be seen from Fig 7.7)

Punch line → 1. Scale invariant power spectrum.

$\Delta^2_{\mathcal{R}}(k) = A_s \Rightarrow$ angle independent (ℓ-indep.) CMB spectrum

2. Amplitude of large scale CMB spectrum is a direct measure of the amplitude ~~of~~ A_s of primordial fluctuations.

7.3.3 Small-Scales: Sound Waves

→ ~~From eqn 7~~ Fig 7.9 shows that $j_\ell(x), j_\ell'(x)$ is peaked near $x \approx \ell$, $x \sim \ell$ (resp.) for large ℓ. Thus for small scales
↳ (less sharp peak)
Integral 7.39 ~~gets~~ can be simplified by taking these terms ($j_\ell(kx)$) as δ-functions.

$$\Rightarrow \quad \frac{\ell(\ell+1)}{2\pi} C_\ell^{SW} \sim F_*^2(k)\, \Delta^2_{\mathcal{R}}(k)\Big|_{k \approx \ell/x_*}$$

$$\frac{\ell(\ell+1)}{2\pi} C_\ell^{D} \sim G_*^2(k)\, \Delta^2_{\mathcal{R}}(k)\Big|_{k \sim \ell/x_*}$$

For a scale ~~indep. + f.~~ invariant initial cond's. $\Delta^2_{\mathcal{R}}(k) = \text{const.}$, the power spectra are determined by $F_*^2(k)$, $G_*^2(k)$ evaluated at $k = \ell/x_*$ & oscillations in k, become oscillations in ℓ.