## (5.2) Collison Terms: Compton Scattering

Having derived eq (5.16) we'll now look at amplitude squared term ($|M|^2$)

$$\sum_{3\text{ spins}} |M|^2 = 32\pi\,\sigma_T m_e^2$$

→ (in deriving this result polarization of radiation field is neglected.)

↳ Thomson cross section

independent of momenta involved

Amplitude does have a polarization dependence which leads to polarization of CMB.

Eq. 3.16

$$C[f(\vec{p})] = \frac{\pi}{8 m_e^2 p}\int \frac{d^3 q}{(2\pi)^3} f_e(\vec{q}) \int \frac{d^3 \vec{p}'}{(2\pi)^3 p'} (32\pi\,\sigma_T m_e^2)$$

$$\times \left\{ \delta_D^{(1)}(p-p') + \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e} \frac{\partial \delta_D^{(1)}(p-p')}{\partial p'} \right\} \left[f(\vec{p}') - f(\vec{p})\right]$$

$$\int \frac{d^3 q}{(2\pi)^3} f_e(\vec{q}) = \frac{n_e}{g_e} = \frac{n_e}{2} \quad \& \quad \underbrace{\int \frac{d^3 q}{(2\pi)^3} f_e(\vec{q}) \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}}_{0} (\;) = \frac{n_e \vec{U}_b}{2}$$

→ (bulk velocity of $e^{\ominus}$)

Also expand $f(\vec{p}')$ & $f(\vec{p})$ as zero order plus perturbation term

$$\Rightarrow \frac{32\pi^2 \sigma_T m_e^2}{8 m_e^2 p}\cdot\frac{n_e}{2}\int \frac{d^3 p'}{(2\pi)^2 p'} \left\{ \delta_D^{(1)}(p-p') + (\vec{p}-\vec{p}')\cdot\vec{U}_b \frac{\partial \delta_D^{(1)}(p-p')}{\partial p'} \right\}$$

$$\times \left[ f^{(0)}(p') - p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\hat{p}') - f^{(0)}(p) + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p}) \right]$$

↳ Note that we are expanding in two small quantities simultaneously, small perturbations $\Theta$ & small energy transfer $(\vec{p}-\vec{p}')$. Here we'll keep only those terms which are first order in either of these small quantities

$$C[f(\vec{p})] = \frac{2\pi^2 n_e \sigma_T}{p} \int_0^\infty \frac{dp'\,(p')^2}{(2\pi)^3\, p'} \int d\Omega' \left[ \delta_D^{(1)}(p-p') \left[ -p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right] + (\hat{p} - \hat{p}')\cdot \vec{U}_b \frac{\partial \delta_D^{(1)}(p-p')}{\partial p'} \left(f^{(0)}(p') - f^{(0)}(p)\right) \right] \quad -(5.19)$$

– Terms containing $\hat{p}'$.

- $\Omega' \rightarrow$ solid $\angle$ spanned by unit vector $\hat{p}'$
- Note that $\Theta$ is taken to be dependent only on $\hat{p}$, $\hat{p}'$ (& not $\vec{x}$, $t$) since collisions are local.

Monopole term $\boxed{\Theta_0(\vec{x}, t) = \frac{1}{4\pi} \int d\Omega' \, \Theta(\hat{p}', \vec{x}, t)}$

↳ fractional perturbation in the angle averaged photon flux at a given position $\vec{x}$ & time $t$.

↳ Will be generalized to a whole sequence of ~~mult~~ multipole moments. $\left\{ \Theta_\ell(k, \eta) = \frac{1}{(-i)^\ell} \int_{-1}^{1} \frac{d\mu}{2} P_\ell(\mu)\, \Theta(\mu, k, \eta) \right\}$

$\because$ $\hat{p}'$ & $\hat{p}$ are independent of $\vec{U}_b$ so $\hat{p}' \cdot \vec{U}_b$ averaged over whole of 3d space gives zero.

Hence integral simplifies to

This term just give since $\int d\Omega'$ is

$$C[f(\vec{p})] = \frac{n_e \sigma_T}{p} \int_0^\infty dp' \cdot p' \left[ \delta_D^{(1)}(p-p') \left[ -p' \frac{\partial f^{(0)}}{\partial p'} \Theta_0 + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right] + \hat{p} \cdot \vec{U}_b \frac{\partial \delta_D^{(1)}(p-p')}{\partial p'} \left( f^{(0)}(p') - f^{(0)}(p) \right) \right]$$

Now we do $p'$ integration

↳ $\int \frac{\partial \delta(n-n')}{\partial n} \cdot f(n) \cdot dn = -\int f'(n)\, \delta(n-n') \cdot dn$ (5.21)

$$C[f(\vec{p})] = \frac{n_e \sigma_T}{p} \left[ p^2 \left( -\frac{\partial f^{(0)}}{\partial p} \Theta_0 + \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right) - p\,\hat{p} \cdot \vec{U}_b \frac{\partial f^{(0)}}{\partial p} \right]$$

$$\boxed{C[f(\vec{p})] = (n_e \sigma_T) \left( -p \frac{\partial f^{(0)}}{\partial p} \right) \left[ \Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{U}_b \right]} \quad -(5.22)$$

~~Someobse~~ observations from $C[f(\vec{p})]$ term

→ ~~In absence of $\vec{U}_b$ ($\vec{U}_b = 0$)~~ the

Pg. 118 - 119 Dodelson .

$\cancel{\Theta_0 = \Theta(x, t)}$ $\Theta(\vec{x}, \hat{p}, t) = \Theta_0(\vec{x}, t)$

→ At strong scattering ~~does~~ inabsence of $\vec{U}_b$

→ If $\vec{U}_b \neq 0$ the photon dist. consists of two terms a monopole & a dipole term implying photons behave like a fluid. Photon & $e^{\ominus}$ behave as a single fluid during strong scattering or tight coupling. Compton scattering ceases to be efficient at photon - baryon decoupling, so photons no longer behave like a fluid after recombⁿ. The "free streaming" phase of photons start after decoupling.

## 5.3 The Boltzmann Eqⁿ for photons

Collecting left & right pieces of boltzmann eqⁿ from (5.9) & (5.22) we get

$$\dot{\Theta} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p}\cdot\vec{U}_b]$$

Replacing physical time $t$ with the conformal time $\eta \to$ (can also be defined as comoving distance of light in absence of any interactions)

$a\, d\eta = dt$

$$\Theta' + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \Phi' + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a [\Theta_0 - \Theta + \hat{p}\cdot\vec{U}_b] \quad - (5.24)$$

where $\dot{\Theta} = \frac{\partial \Theta}{\partial t}$ & $\Theta' = \frac{\partial \Theta}{\partial \eta}$

## Utility of Fourier Space

Consider a field $\delta(\vec{x}, t)$ obeying linear PDE

$$\frac{\partial^2 \delta(\vec{x},t)}{\partial t^2} + f(t)\frac{\partial \delta(\vec{x},t)}{\partial t} + g(t)\nabla^2\varphi = 0 \qquad - (5.25)$$

Note that the coeffs $f, g$ are only functions of time coz ~~we~~ ~~was~~ the only $\vec{x}$ dependence is due to perturbation & we work in linear order in them.

Spatial Fourier transform

$$\delta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\delta}(\vec{k})$$

$$\Leftrightarrow \quad \tilde{\delta}(\vec{k}) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \delta(\vec{x})$$

$$\Rightarrow \frac{\partial \delta(\vec{x},t)}{\partial x^i} = \int \frac{d^3k}{2\pi} e^{i\vec{k}\cdot\vec{x}}\left(ik_i \tilde{\delta}(\vec{k})\right)$$

$$\Rightarrow \frac{\partial \delta(\vec{x},t)}{\partial x^i} \xleftrightarrow[(F.T)]{} ik_i \tilde{\delta}(\vec{k},t)$$

__Note__: $k_i, \partial_i, v_{b_i}$ are all 3D ~~ve~~ euclidean vector comps.

hence $k_i = k^i$, $\partial_i = \partial^i$, $v_{b_i} = v_b^i$

We can drop ~~a~~ ~ from fourier transformed quantities (as their argument make it obvious) $\tilde{\delta}(\vec{k},t) \to \delta(\vec{k},t)$

Eq$^n$ (5.25) gets transformed to

PDE
$$\frac{\partial^2 \delta(\vec{k},t)}{\partial t^2} + f(t)\frac{\partial \delta(\vec{k},t)}{\partial t} \mp g(t)k^2\varphi(\vec{k},t) = 0$$

ODE
$$\boxed{\frac{d^2\delta(t)}{dt^2} + f(t)\frac{d\delta(t)}{dt} - g(t)k^2\varphi(t) = 0}$$

~~The eq$^n$ can be~~

→ The eq$^n$ can be solved independently for each $\vec{k}$ without knowing ~~st~~ sol$^n$ for other values of $\vec{k}'$. (A feature which was absent in eq$^n$ 3.25). ⇒ "Every Fourier Mode evolves independently"

→ We'll go back to solving eq$^n$ (5.24) in fourier space but first look at two new useful relations.

(i) $\boxed{\mu = \dfrac{\vec{k}\cdot\hat{p}}{k}}$ → cosine of angle b/w wavenumber $\vec{k}$ & photon direction $\hat{p}$.

Note that the wave vector $\vec{k}$ points in the direction in which temperature is changing. (i.e. it is parallel to gradient)

eg. $\Theta(k, \mu = 1)$ ↓ photons travelling along the dir$^n$ where temp. is changing

( ~~$\nabla \Theta(\vec{x}, t) \longleftrightarrow \vec{k}\,\Theta(\vec{k}, t)$~~ )

( $\nabla\,\Theta(\vec{x}, t) \longleftrightarrow i\vec{k}\,\Theta(\vec{k}, t)$ )

$\Theta(k, \mu = 0)$ → ― ― ― ― $\perp$ to gradient

• (ii) In cosmology velocities usually point in same direction as $\vec{k}$ (longitudinal)

$$\vec{v}_b(\vec{k}, \eta) = \frac{\vec{k}}{k}\, v_b(\vec{k}, \eta)$$

$$\Rightarrow \vec{v}_b \cdot \hat{p} = v_b \cdot \mu$$

Optical Depth $\quad \tau(\eta) = \displaystyle\int_{\eta}^{\eta_0} d\eta' (n_e \sigma_T a)$ at late times (recent times) $n_e$ is very small hence $\tau$ ~~(..)~~ $<< 1$ while at earlier times $n_e$ is quite high.

$$\tau' = \frac{d\tau}{d\eta} = -n_e \sigma_T a$$

Combining all these eq$^n$ (5.24) simplifies to

$$\boxed{\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'\left[\Theta_0 - \Theta + \mu v_b\right]} \quad -(5.35)$$

↳ diff. fourier modes $\vec{k}$ are decoupled. We can solve for each → $\vec{k}$ & $\mu$ indep

## 5.4 The Boltzmann Equation for Cold Dark Matter (CDM)

→ Just like photons here also the starting point to describe evolution of dark matter is Boltzmann eqn.

→ The main differences that arises in dark matter dist. are due to

- At epochs long after detection dark matter doesn't interact with any other particles hence collision terms are zero.
- CDM is non relativistic $(v << c)$

We use Boltzmann eq$^n$ for massive particles (eq 3.76)

$$\frac{\partial f_c}{\partial t} + \left(\frac{p}{E}\right)\frac{\hat{p}^i}{a}\frac{\partial f_c}{\partial x^i} - \left[H + \dot{\Phi} + \frac{E}{ap}\hat{p}^i\frac{\partial \Psi}{\partial x^i}\right]p\frac{\partial f_c}{\partial p} = 0 \qquad - (5.36)$$

$(p/E = v)$ → these velocity factors supress "free streaming"

→ In case of relativistic particle (photons) we assumed a form of $f^{(0)}$ (BE dist.) & considered linear order perturbations around it.

Here instead we'll start ~~to~~ by taking moments of eq$^n$ (3.76) & use the fact dark matter particles are "very non-relativstic" implying terms of order ~~$(p/m)^2$ &~~ higher than $(p/m)$ can be neglected.

multiplying 5.36 by $\frac{d^3p}{(2\pi)^3}$ & integrating

$$\frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3}f_c + \frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}f_c\frac{p\hat{p}^i}{E(p)} - (H+\dot{\Phi})\int\frac{d^3p}{(2\pi)^3}p\frac{\partial f_c}{\partial p}$$

$$-\frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial p}E(p)\,\hat{p}^i = 0 \qquad - (5.37)$$

$$\int\frac{d^3p}{(2\pi)^3}f_c = n_c \qquad \int\frac{d^3p}{(2\pi)^3}f_c\frac{p\hat{p}^i}{E(p)} = n_c v^i$$

3rd term $\int\frac{d^3p}{(2\pi)^3}p\frac{\partial f_c}{\partial p} = \frac{1}{(2\pi)^3}\int dp\cdot p^3\int d\Omega\frac{\partial f_c}{\partial p} = \frac{1}{(2\pi)^3}\int dp\cdot p^3\frac{\partial}{\partial p}\int d\Omega f_c$

$$= -3\int\frac{d^3p}{(2\pi)^3}f_c = -3n_c \leftarrow \quad = \frac{1}{(2\pi)^3}\left[p^3\int d\Omega f_c\Big|_0^\infty - 3\int_0^\infty dp\,p^2\int d\Omega f_c\right]$$

By parts to the 4th part

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial f_c}{\partial p} B(p)\, \hat{p}^i = \int \frac{dp \cdot p^2}{(2\pi)^3} \frac{\partial}{\partial p} \int f_c\, m\, \hat{p}^i\, d\Omega$$

$$= \frac{p^2}{(2\pi)^3} \underbrace{\int f_c\, m\, \hat{p}^i\, d\Omega \Big|_0^\infty}_{} - \int \frac{2\,dp\, p}{(2\pi)^2} \int f_c\, m\, \hat{p}^i\, d\Omega$$

$f_c$ integrated over $p \to \infty$ surface is 0.

~~kind~~ ⓑ

This term goes to 0 (??)

ⓑ How to ensure it approaches 0 faster than $p^2$ (??)

So eqⁿ (5.37) finally becomes

$$\underbrace{\frac{\partial n_c}{\partial t} + \frac{1}{a} \frac{\partial (n_c v_c^i)}{\partial x^i}}_{\text{Continuity eq}^n} + 3\underbrace{[H + \dot{\Phi}]}_{\substack{\text{local perturbed} \\ \text{hubble rate}}} n_c = 0 \quad - \quad (5.41)$$

We separate (5.41) into a zero order & first order piece.

$v_c$ & $\Phi$ are first order terms.

$$\frac{\partial \bar{n}_c}{\partial t} + 3H \bar{n}_c = 0 \rightarrow \bar{n}_c \text{ zeroth order homogenous part of density}$$

$$\Rightarrow \frac{d}{dt}(\bar{n}_c a^3) = 0 \Rightarrow \bar{n}_c \propto a^{-3}.$$

First order part → we'll set $n_c(\vec{x}, t) = \bar{n}_c(t)[1 + \delta_c(\vec{x}, t)]$

energy density ← $\rho_c(\vec{x}, t) = \bar{\rho}_c(t)[1 + \delta_c(\vec{x}, t)] = m n_c(\vec{x}, t)$

$\delta_c = \frac{\delta \rho_c}{\bar{\rho}_c}$ → fractional overall density

$$\cancel{\frac{\partial (\delta_c(\vec{x},t)\vec{x})}{\partial t} + \frac{1}{a} \frac{\partial (v_c^i)}{\partial x^i} + 3\dot{\Phi}}$$

↳ ~~First order eqⁿ after dividing by $\bar{n}_c(t)$~~

$$\frac{\partial (\bar{n}_c(t) \delta_c(\vec{x}, t))}{\partial t} + \frac{1}{a} \frac{\partial (v_c^i)\, \bar{n}_c(t)}{\partial x^i} + 3H \bar{n}_c(t) \delta_c(\vec{x}, t) + 3\dot{\Phi}\, \bar{n}_c(t) \delta_c(\vec{x}, t) = 0$$

dividing by $\bar{n}_c$ we get $\boxed{\frac{\partial \delta_c}{\partial t} + \frac{1}{a} \frac{\partial v_c^i}{\partial x^i} + 3\dot{\Phi} = 0}$

We've introduced two new perturbation variables for the dark matter, the density perturbation $\delta_c$ & velocity $u_c$. We'll need one more eq$^n$ besides (5.45). One more eq$^n$ is obtained by taking first moment of eq$^n$ (5.36).

$$\underbrace{\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f_c \frac{p\hat{p}^j}{E}}_{n_c u_c^j} + \underbrace{\frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3}\frac{p^3}{E^2}\hat{p}^i\hat{p}^j}_{\text{Can be neglected due to } p^2/E^2} - (H+\dot{\Phi})\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{E}\frac{\partial f_c}{\partial p}\hat{p}_j$$

$$\left(\frac{p^j}{m}\right)$$

$$-\frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} p\frac{\partial f_c}{\partial p}\hat{p}^i\hat{p}^j$$

3rd term $\rightarrow$

$$\int \frac{d\Omega}{(2\pi)^3}\hat{p}_j\int_0^\infty \frac{p^4}{E}\frac{\partial f_c}{\partial p}\,dp = \int \frac{d\Omega}{(2\pi)^3}\hat{p}_j\left[\cancel{\frac{p^4 f_c}{E}\Big|_0^\infty} - \int_0^\infty f_c\left(\frac{4p^3}{E} - \frac{p^4}{E^2}\frac{dE}{dp}\right)\right]$$

How to ensure $f_c$ falls faster ($E^2 = p^2 + m^2$)

$$= \int \frac{d\Omega}{(2\pi)^3}\hat{p}_j\int_0^\infty -f_c\left(\frac{4p^3}{E} - \frac{p^5}{E^3}\right).dp$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{-4f_c p\hat{p}_j}{E}\right] + \cancel{\left(\frac{f_c p\hat{p}_j}{E}\right)}\left(\frac{p^2}{E^2}\right)$$

$(p^3/E^3)$

$$= -4n_c u_c^j$$

4th term $\rightarrow$ ① $\int d\Omega\,\hat{p}^i\hat{p}^j = \frac{4\pi}{3}\delta_{ij}$ (a/c to book) $\cdot$ ~~$4\pi\,\delta_{ij}$ (a/c to me)~~ (also to be consistent with eq$^n$ (5.49))

$$\int \frac{d^3p}{(2\pi)^3}p\frac{\partial f_c}{\partial p}\hat{p}^i\hat{p}^j = \int \frac{d\Omega\,\hat{p}^i\hat{p}^j}{(2\pi)^3}\int_0^\infty p^3\frac{\partial f_c}{\partial p}.dp = \int \frac{d\Omega}{(2\pi)^2}\hat{p}^i\hat{p}^j\left[p^3 f_c\Big|_0^\infty - \int_0^\infty 3p^2 f_c.dp\right]$$

$$= -n_c\delta_{ij} \rightarrow ①$$

Finally we obtain

$$\frac{\partial}{\partial t}(n_c u_c^j) + 4Hn_c u_c^j + \frac{n_c}{a}\frac{\partial \Psi}{\partial x^j} = 0 \quad -(5.49)$$

The eq$^n$ has no zero order parts. We only need to extr out first order parts.

$$u_c^j\frac{\partial \bar{n}_c}{\partial t} + \bar{n}_c\frac{\partial u_c^j}{\partial t} + 4H\bar{n}_c u_c^j + \frac{\bar{n}_c}{a}\frac{\partial \Psi}{\partial x^j} = 0 \Rightarrow \boxed{\frac{\partial u_c^j}{\partial t} + Hu_c^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = 0} \;(5.50)$$

→ We express eqⁿ's (5.45) & (5.50) in terms of conformal time η & in fourier space.

Eqⁿ (5.45) → $\frac{\partial \delta_c}{\partial t} + \frac{1}{a}\frac{\partial v_c^i}{\partial x^i} + 3\frac{d\Phi}{dt} = 0$

$$\frac{\partial \delta_c}{a\,\partial \eta} + \frac{1}{a} i k_i v_c^i + \frac{3\,d\Phi}{a\,d\eta} = 0$$

Here also we take velocity $v_c$ in direction of $\vec{k}$

i.e. $v_c^i = \frac{k^i}{k} v_c$

$$\Rightarrow \quad k_i v_c^i = \frac{k_i k^i}{k} v_c = k v_c$$

$$\Rightarrow \boxed{\delta_c' + i k v_c + 3\Phi' = 0} \quad -(5.51)$$

Similarly 5.50 gets transformed to

$$\boxed{v_c' + \frac{a'}{a} v_c + i k \Psi = 0} \quad -(5.52)$$

## The Boltzmann Eqⁿ for Baryons

→ whenever we'll speak of baryons we'll only mean proton

⇒ $e^{\ominus}+p \rightarrow$ Baryons (Misnomer) → coupled by Coulumb scattering $\underbrace{(e^{\ominus}+p \leftrightarrow e^{\ominus}+p)}$ · Very tightly coupled

Can be assumed to be coupled at all epochs (Scattering rate that high !!) leads to

$$\frac{\rho_e-\bar{\rho}_e}{\bar{\rho}_e} = \frac{\rho_p-\bar{\rho}_p}{\bar{\rho}_p} \equiv \delta_b \rightarrow \text{(common value of overdensities)}$$

$\delta_b$ ↓ Baryons

Similarly $\vec{U}_e = \vec{U}_p \equiv \vec{U}_b$ → Baryons

After recombⁿ when $e^{\ominus}$ & N (nuclei) first form atoms, this tight coupling remains while the neutral atoms are now decoupled from photons. But free $e^{\ominus}$ are still coupled to photons via Compton scattering. At epochs around recombⁿ $T \ll m_e$ hence $e^{\ominus}$ & N can be taken to be non relativistic fluid & hence just like CDM case we'll ~~take~~ consider first two moments of botzmann eqⁿ.

Zeroth moment eqⁿ (5.51) → $\boxed{\delta_b' + ikU_b + 3\Phi' = 0}$

(At epochs around recombⁿ, the reactions which change no. like annihilations, pair prodⁿ & nuclear rxns are irrelevant)

Note that we've put 0 in RHS coz zero moment eqⁿ is actually a number conservation eqⁿ ($e^{\ominus}$ in this case) so the Collision terms (which appear in RHS) don't affect it. ↳ They just correspond to those reactions which are actually scatterings (Columbs / compton)

first moment eqⁿ for baryons → first moment BE is actually a momentum conservation eqⁿ. ~~We add~~ Second eqⁿ for baryons is obtained by taking first moment BE for $e^{\ominus}$ & baryon & adding them. Note that earlier we took moments by multiplying $\frac{\vec{p}}{E}$ but now ~~we~~ we'll multiply only $\vec{p}$. (Basically multiply on individual masses m to eqⁿ (5.49) & adding). In RHS $m_p$ dominates coz $m_p \gg m_e$

$$\Rightarrow m_p \frac{\partial(n_b U_b^j)}{\partial t} + 4H m_p n_b U_b^j + \frac{m_b n_b}{a}\frac{\partial \varphi}{\partial x^j} = F_{e\gamma}^j(\vec{x}, t) \qquad (5.56)$$

↓ This ~~term~~ time RHS is not zero cos momentum of ~~pho~~ $e^{\ominus}$ + baryons is not conserved. ~~$e^{\ominus}$~~ as photons transfer mom. ~~this~~ to $e^{\ominus}$ through compton scattering.

Dividing both sides of (5.56) by $\rho_b = m_b \bar{n}_b$

$$\frac{\partial u_b^j}{\partial t} + H u_b^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = \frac{1}{\rho_b} F_{er}^i(\vec{x}, t)$$

Since momentum is conserved in each scattering event this force term (~~momentum~~ rate of change of tot. mom.) has to be equal & opposite to the force term appearing in "photon analog of Baryon Euler Eq^n".

→ Assuming that the direction of $\vec{F}_{er}$ term would be along wavevector $\vec{k}$ (also the direction ~~in which photon~~ of Temp. gradient). Multiplying by $\hat{k}_i$ before taking first moment.

negative of photon collison term

$$\frac{1}{\rho_b}\hat{k}_i F_{er}^i(\vec{x},t) = \ominus 2 \frac{n_e \sigma_T}{\rho_b}\int \frac{d^3p}{(2\pi)^3}\, \vec{p}\hat{k}_i\left[-p\frac{\partial f^{(0)}}{\partial p}\right]\left[\Theta_0 - \Theta(\mu) + \mu u_b\right] \quad (5.58a)$$

As momentum neve' of $e^{\ominus}$ counts both spin states

↳ Fourier space version of eq^n (5.22)

$$p\hat{p}^i \cdot \hat{k}_i = p\mu$$

$$d^3p = dp\, p^2 \sin\theta\, d\theta\, d\phi = (p^2 dp)(d\mu)(2\pi) \quad \text{(assuming dir}^n \text{ of wavevector } \vec{k} \text{ as } \hat{z} \text{ axis)}$$

$$(\cos\theta = \mu)$$

$$\frac{1}{\rho_b}\hat{k}_i F_{er}^i(\vec{x},t) = \frac{2 n_e \sigma_T}{\rho_b}\underbrace{\int \frac{dp\, p^4}{2\pi^2}\frac{\partial f^{(0)}}{\partial p}}\int_{-1}^{1}\frac{d\mu}{2}\mu\left[\Theta_0 - \Theta(\mu) + \mu u_b\right]$$

ind. of $\mu$ hence this integral evalues to 0

ind. of $\mu$ Evaulate to $\frac{u_b}{3}$

$$p^4 f^0\Big|_0^{\infty} - \int_0^{\infty}\frac{dp}{2\pi^2}\cdot 4p^3 f^{(0)} = -2\rho_r \quad \text{(eq 2.73)}$$

Consider the integral $\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta(\mu)$

We earlier defined monopole term as $\Theta_0(\vec{x},t) \equiv \frac{1}{4\pi}\int d\Omega' \,\Theta(\hat{p}',\vec{x},t)$

$$\Theta_0(k,\eta) = \frac{1}{4\pi}\int_{-1}^{1} 4\pi \frac{d\mu}{2}\,\Theta(\mu)$$

So it makes sense to define the dipole term as

$$i\int_{-1}^{1}\frac{d\mu}{2}\mu\,\Theta(\mu,k,\eta) = \Theta_1(k,\eta)$$

from convention

Eq^n (5.58) becomes

$$\boxed{\frac{1}{\rho_b}\hat{k}_i F_{er}^i(\vec{x},t) = -n_e\sigma_T\frac{4\rho_r}{\rho_b}\left[i\Theta_1 + \frac{1}{3}u_b\right]} \quad (5.59)$$

For a nice interpretation of dipole term refer Dodelson para after (5.60)

Eq$^n$ (5.57) after switching to conformal time becomes

$$\boxed{u_b' + \frac{a'}{a} u_b + ik\Psi = \tau' \frac{4\rho_\gamma}{3\rho_b}\left[3i\Theta_1 + u_b\right]}$$

Nice discussion after 5.61 as well

## 5.6 The Boltzmann Eq$^n$ for Neutrinos

For neutrinos we can proceed with an analysis just like photons bcoz neutrinos possess an equilibrium dist. ( fermi- ~~d.f.~~ dirac d.f.) ~~if they as they do~~

$$f_\nu(\vec{n}, \vec{p}, t) = \left[\exp\left\{\frac{p}{T_\nu(t)\{1+N(\vec{n}, \vec{p}, t)\}}\right\} + 1\right]^{-1}$$

Note that unlike $\Theta(\vec{n}, \hat{p}, t)$, $N$ also depends on magnitude of $\vec{p}$ (as the momentum keeps on decreasing as universe expands)

$$= \left[1 - N(\vec{n}, \vec{p}, t)\, p\frac{d}{dp}\right] f_\nu^{(0)}(p) \quad (5.62)$$

$f_\nu^{(0)}(p) = \left[e^{p/T_\nu(a)} + 1\right]^{-1} \rightarrow$ zeroth order d.f. $\rightarrow$ After decoupling

Epochs of interest $\rightarrow$ Decoupling onwards. $\rightarrow$ neutrinos don't ~~have~~ interact with any particles so apt ~~would be~~ for massive particles. eq$^n$ would be collisionless boltzmann eq$^n$

$$\frac{df_\nu}{dt} = \frac{\partial f_\nu}{\partial t} + \frac{p}{E_\nu(p)}\frac{\hat{p}^i}{a}\frac{\partial f_\nu}{\partial n^i} - \left[H + \dot{\Phi} + \frac{E_\nu(p)}{ap}\hat{p}^i\frac{\partial \Psi}{\partial n^i}\right] p\frac{\partial f_\nu}{\partial p} = 0$$

Inserting (5.62), at zero order we obtain homogenous boltzmann eq$^n$, at first order we obtain

$$\cancel{\frac{df_\nu}{dt}} \cancel{-f_\nu(p)\frac{\partial N}{\partial t}}$$

$$-\frac{pN\partial^2 f^{(0)}}{\partial t\,\partial p} - \frac{p\, df_\nu^{(0)}}{dp}\frac{\partial N}{\partial t} - \frac{p}{E_\nu(p)}\frac{\hat{p}^i}{a}\frac{\partial N}{\partial n^i}\frac{p\, df_\nu^{(0)}(p)}{dp} - Hp\frac{\partial}{\partial p}\left[-p\frac{\partial f_\nu^{(0)}}{\partial p}N\right]$$

$$-p\frac{\partial f_\nu^{(0)}}{\partial p}\left[\dot{\Phi} + \frac{\hat{p}^i}{a}\frac{E_\nu(p)}{p}\frac{\partial \Psi}{\partial n^i}\right] = 0$$

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{p}{E_\nu(p)} \frac{\hat{p}^i}{a} \frac{\partial \mathcal{N}}{\partial x^i} - Hp \frac{\partial \mathcal{N}}{\partial p} + \dot{\Phi} + \frac{E_\nu(p)}{ap} \hat{p}^i \frac{\partial \Psi}{\partial x^i} = 0 \quad - (5.67)$$

Converting time derivs to conformal time derivs & changing to fourier space we get

$$\mathcal{N}'(\vec{k}, p, \mu, \eta) + i\, k\mu \frac{p}{E_\nu(p)} \mathcal{N} - Hp \frac{\partial \mathcal{N}}{\partial p} = -\Phi' - ik\mu \frac{E_\nu(p)}{p} \Psi$$

Once neutrinos are no longer relativistic ~~their~~ the distribution can be very different for neutrinos in low energy tail of the dist. than for those in high energy tail.

→ if we are only dealing with behaviour of neutrinos upto recombination, then we can set $\frac{p}{E_\nu(p)} = 1$ & neglect $p$ dependence of $\mathcal{N}$, getting a collisionless version of photon eqn.