Ch-6 The inhomogenous Universe : Gravity

→ 6.1 Scalar Vector Tensor Decomposition

We break perturbations as

$$g_{00}(t,\vec{x}) = -1 + h_{00}(t,\vec{x})$$

$$g_{0i}(t,\vec{x}) = a(t)\, h_{0i}(t,\vec{x}) - h_{i0}(t,\vec{x})$$

$$g_{ij}(t,\vec{x}) = a^2(t)\left[\delta_{ij} + h_{ij}(t,\vec{x})\right]$$

We want to classify the components of general metric via their behaviour under spatial rotations.

→ h_{00} component → 3 scalar → $\boxed{h_{00} = -2A}$

→ h_{0i} → 3vector → $\boxed{h_{0i} = -\frac{\partial B}{\partial x^i} - B_i}$; $\nabla\cdot\vec{B} = 0 = \frac{\partial B_i}{\partial x^i} = 0$

$B \to$ scalar contri $\vec{B} \to$ vector contri

$$\boxed{h_{0i}(t,\vec{k}) = -ik_i B(t,\vec{k}) - B_i(t,\vec{k})}$$

$\boxed{k^i B_i = 0}$

(Fourier Space)

→ $\boxed{h_{ij} = \underbrace{2D\delta_{ij} - 2E_{ij}}_{\text{Scalar}} + \underbrace{V_{i,j} + V_{j,i}}_{\text{Vector}} + \underbrace{h_{ij}^{TT}}_{\text{Tensor}}}$; $\nabla\cdot\vec{V} = 0$

$\boxed{h_{ij} = 2D\delta_{ij} + 2k_i k_j E + ik_i V_j + ik_j V_i + h_{ij}^{TT}}$

→ $h_{\mu\nu}$ → 10 dof → $\underbrace{(A, B, D, E)}_{4}$, $\underbrace{(\vec{B}, \vec{V})}_{2+2}$, $\underbrace{h_{ij}^{TT}}_{2}$

→ Decomposition Theorem : Perturbations of each type (scalar, vector, tensor) evolve independently at linear order. Eg. if a physical process sets up tensor perturb. in early universe, they do not induce scalar perturbations.

→ 6.2 From gauge to gauge

Q: While dealing with scalar perturbations, why didn't we consider B_i, E_{ij} in h_{0i} & h_{ij}.
Ans: This section.

→ In context of perturbation theory in relativity, a choice of coordinates referred to as gauge.

→ Consider a scalar field

$$\varphi(\vec{n},t) = \bar{\varphi}(t) + \delta\varphi(t,\vec{n})$$

↳ bg field; only depend on time as universe is homogenous

We do a coord. transf.

$$n \to \hat{n} \; ; \quad \boxed{\begin{aligned} t &\to \hat{t} = t + \varsigma(t,\vec{n}) \\ n^i &\to \hat{n}^i = n^i + \xi^{,i}(t,\vec{n}) \end{aligned}} \qquad - (6.8)$$

↳ written as gradient, since we are only considering scalar perturbation

Since φ is scalar

$$\varphi(n) = \hat{\varphi}(\hat{n}) = \hat{\varphi}(\hat{t},\vec{\hat{n}})$$

$$\hat{\varphi}(\hat{t},\vec{\hat{n}}) = \varphi(t,\vec{n}) = \varphi(\hat{t} - \varsigma(t,\vec{n}), \vec{\hat{n}} - \nabla\xi)$$

$$= \varphi(\hat{t},\vec{\hat{n}}) + \left.\frac{\partial\varphi}{\partial\hat{t}}\right|_{\hat{t},\hat{n}} (-\varsigma(t,\vec{n}))$$

$$+ \frac{\partial\varphi}{\partial\hat{n}^i}(-\xi^{,i}(t,\vec{n}))$$

Produce a 2nd order quantity

$$\hat{\varphi}(\hat{t},\hat{n}) = \varphi(\hat{t},\vec{\hat{n}}) - \frac{d\bar{\varphi}(\hat{t})}{d\hat{t}}\varsigma(\hat{t},\hat{n})$$

$$= \bar{\varphi}(\hat{t}) + \underbrace{\delta\varphi(\hat{t},\vec{\hat{n}}) - \frac{d\bar{\varphi}(\hat{t})}{d\hat{t}}\varsigma(\hat{t},\vec{\hat{n}})}$$

$$\Rightarrow \quad \delta\hat{\varphi}(\hat{t},\vec{\hat{n}}) = \delta\varphi(\hat{t},\vec{\hat{n}}) - \frac{d\bar{\varphi}(\hat{t})}{d\hat{t}}\varsigma(\hat{t},\vec{\hat{n}})$$

$$\boxed{\delta\hat{\varphi}(t,\vec{n}) = \delta\varphi(t,\vec{n}) - \frac{d\bar{\varphi}(t)}{dt}\varsigma(t,\vec{n})} \quad \to (\text{Sol}^n \text{ Eq. } 6.3)$$

→ We use this coordinate transf. (6.8) to the general perturbed metric with scalar perturbations.

$$g_{00} = -(1+2A)$$

$$g_{0i} = -aB_{,i}$$

$$g_{ij} = a^2(\delta_{ij}[1+2D] - 2E_{,ij})$$

Using $\hat{g}_{\alpha\beta}(\hat{x}) \cdot \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \cdot \frac{\partial \hat{x}^\beta}{\partial x^\nu} = g_{\mu\nu}(x)$

(i) $\mu = 0, \nu = 0$

$$\hat{g}_{\alpha\beta}(\hat{x}) \cdot \frac{\partial \hat{x}^\alpha}{\partial t} \cdot \frac{\partial \hat{x}^\beta}{\partial t} = g_{00}(x) = -(1+2A)$$

→ if $\alpha = i, \beta = j$

$\frac{\partial \hat{x}^i}{\partial t} \cdot \frac{\partial \hat{x}^j}{\partial t} \approx (\xi)^2$ → 2nd order

if $\alpha = 0, \beta = i$

$\hat{g}_{0i}(\hat{x}) \frac{\partial \hat{x}^0}{\partial t} \cdot \frac{\partial \hat{x}^i}{\partial t} \sim (B\varepsilon_a)$ → 2nd order

Hence $\hat{g}_{00}(\hat{x}) \left(\frac{\partial \hat{t}}{\partial t}\right)^2 = -(1+2A)$

$$-(1+2\hat{A})(1+\dot{\xi})^2 = -(1+2A)$$

$$-2\hat{A} - 2\dot{\xi} = -2A$$

$\Rightarrow \boxed{A \rightarrow \hat{A} = A - \dot{\xi} = A - \frac{\xi'}{a}}$; $\xi' = \frac{d\xi}{d\eta} = a\frac{d\xi}{dt}$

(ii) $\mu = 0, \nu = i$

$$\hat{g}_{\alpha\beta}(\hat{x}) \cdot \frac{\partial \hat{x}^\alpha}{\partial t} \cdot \frac{\partial \hat{x}^\beta}{\partial x^i} = g_{0i}(x) = -a\frac{\partial B}{\partial x^i}$$

$$-(1+2\hat{A})(1+\dot{\xi})(\xi^i) + (-a\hat{B}_{,i})(1+\dot{\xi})\left(1+\frac{\partial^2 \varepsilon_a}{\partial x^{i2}}\right) + (a^2 \delta_{ij})\left(\frac{\partial^2 \varepsilon_a}{\partial t\, \partial x^i}\right)\left(1+\frac{\partial^2 \varepsilon_a}{\partial x^{i2}}\right)$$

$$= -a B_{,i}$$

$$-\xi^i - a\hat{B}_{,i} + a\varepsilon'_{a,i} = -aB_{,i}$$

$$\boxed{\hat{B} = B + \varepsilon'_a - \frac{\xi}{a}}$$

(iii) $\mu = i \quad \nu = j$

$$\hat{g}_{\alpha\beta}(\hat{x}) \frac{\partial \hat{x}^{\alpha}}{\partial x^i} \cdot \frac{\partial \hat{x}^{\beta}}{\partial x^j} = g_{ij}(x) = a^2 \left(\delta_{ij} \left[1 + 2D \right] - 2E_{,ij} \right)$$

$$(-a B_{,i}) \left(1 + \frac{\partial^2 \mathcal{E}_s}{\partial x^{i2}}\right) \left(\frac{\partial \xi}{\partial x^j}\right) + a^2 \left(\delta_{\hat{k}\hat{l}} \left[1 + 2\hat{D} \right] - 2 \hat{E}_{\hat{k}\hat{l}} \right) \left(\delta^k_i + \frac{\partial^2 \mathcal{E}}{\partial x^i \partial x^k} \right) \left(\delta^l_j + \frac{\partial^2 \mathcal{E}}{\partial x^j \partial x^l} \right)$$

$$= a^2 \left(\delta_{ij} \left[1 + 2D \right] - 2E_{,ij} \right)$$

$$\underbrace{a^2 \delta_{ij} \left[1 + 2\hat{D} \right] + a^2 \delta_{kj} \frac{\partial^2 \mathcal{E}}{\partial x^i \partial x^k} + a^2 \delta_{il} \frac{\partial^2 \mathcal{E}}{\partial x^j \partial x^l}}_{a^2 \mathcal{E}_{,ij} + a^2 \mathcal{E}_{,ij}} - 2\hat{E}_{,ij} = a^2 \left(\delta_{ij} \left[1 + 2D \right] - 2E_{,ij} \right)$$

$$\Rightarrow \boxed{\hat{E}_{ij} = E_{,ij} + \mathcal{E}_{,ij}} - \boxed{\hat{E} = E + \mathcal{E}}$$

$$\boxed{\hat{D} = D - H\xi} - ??$$

→ These transformation relations are just functions of 2 ind. functions ξ, $\mathcal{E}$. Thus there are 2 degrees of freedom

→ In conformal-Newtonian Gauge we choose B & E to be zero.

→ ~~In one - another gauge~~, We can consider linear comb^n of metric perturbations (A, B, D, E) that are invariant under above transformation

$$\boxed{\begin{aligned} \Phi_A &= A + \frac{1}{a} \frac{\partial}{\partial \eta} \left[a (E' - B) \right] \\ \Phi_H &= -D + aH(B - E') \end{aligned}}$$

→ Gauge-invariant variables

→ In conformal newtonian-gauge $A = \psi$, $B = 0$, $D = \Phi$, $E = 0 \Rightarrow \Phi_A = \psi$, $\Phi_H = -\Phi$

→ Corresponding perturbative terms in $T_{\mu\nu}$ are no density perturbation δ_s & longitudnal velocity v_s.

→ We can also introduce a vector coordinate transf. $\mathcal{E}_i$ & use this to set one of B_i or V_i to zero. This will reduce vector perturb from four to two dof. Tensor perturb is unaffected by coord transf. So in total we have 6 dof.

(2 scalar + 2 vector + 2 tensor)

6.3 The Einstein equations for scalar perturbations

→ We work out Einstein's eqⁿ at linear order. To begin we'll focus on scalar perturbation & continue to work in conformal - ~~Ne~~ Newtonian gauge.

$$g_{00}(\vec{x},t) = -1 - 2\Psi(\vec{x},t)$$
$$g_{0i}(\vec{x},t) = 0$$
$$g_{ij}(\vec{x},t) = a^2(t)\,\delta_{ij}\,[1 + 2\Phi(\vec{x},t)]$$

Ricci Tensor

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} \cancel{- \Gamma^{\alpha}} \quad \cancel{g^{\alpha\beta} R_{0\alpha 0\beta} = g^{\alpha\beta} g_{0\gamma} R^{\gamma}_{\alpha 0 \beta}} \quad R^{\alpha}_{0\alpha 0}$$
$$= \cancel{g^{\alpha\beta}\left[g_{0\gamma}\right]\left[\partial_0 \Gamma^{\gamma}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma}_{\alpha 0} + \Gamma^{\gamma}_{0h}\Gamma^{h}_{\alpha\beta} - \Gamma^{\gamma}_{\beta h}\Gamma^{h}_{\alpha 0}\right]}$$
$$= \partial_\alpha \Gamma^{\alpha}_{00} - \partial_0 \Gamma^{\alpha}_{\alpha 0} + \Gamma^{\alpha}_{\alpha h}\Gamma^{h}_{00} - \Gamma^{\alpha}_{0h}\Gamma^{h}_{\alpha 0}$$

first term → $\partial_0 \Gamma^0_{00} = \ddot{\Psi}$

$$\partial_i \Gamma^i_{00} = \frac{1}{a^2}\Psi_{,ii} = \frac{-\sum_i k_i^2 \Psi}{a^2} = -\frac{k^2\Psi}{a^2}$$

2nd term → $\partial_0 \Gamma^0_{00} = \ddot{\Psi}$

$$\partial_0 \Gamma^i_{i0} = 3(\partial_0 H + \ddot{\Phi})$$
$$\downarrow$$
$$\left(\frac{\ddot{a}}{a} - H^2\right)$$

3rd term $\Gamma^0_{0h}\Gamma^h_{00} = (\dot{\Psi})^2 + \frac{(ik_i\Psi)(ik_i\Psi)}{a^2}$

$$= (\dot{\Psi})^2 - \frac{k^2\Psi^2}{a^2}$$

$$\Gamma^i_{ih}\Gamma^h_{00} = 3(H + \dot{\Phi})\,\cancel{(ik_i\Psi)}(\dot{\Psi}) + \frac{(ik_k\Psi)(ik_k\Psi)}{a^2}$$

4th term → $\Gamma^0_{0h}\cdot\Gamma^h_{00} = (\dot{\Psi})^2 + -\frac{k^2\Psi^2}{a^2}$

$$\cancel{\Gamma^0_{0i}\Gamma^i_{00} = \frac{(ik_i\Psi)(ik_i\Psi)}{a^2} = -\frac{k^2\Psi^2}{a^2}}$$

$$\Gamma^i_{0h}\Gamma^h_{i0} = \left(\frac{ik_i\Psi\, ik_i\Psi}{a^2}\right) + \sum_{i,k}(\delta_{ik})^2(H + \dot{\Phi})^2 = \frac{-k^2\Psi^2}{a^2} + 3(H^2 + 2H\dot{\Phi})$$

→ Terms of 2nd order

$$\boxed{R_{00} = -3\frac{\ddot{a}}{a} - \frac{k^2}{a^2}\Psi - 3\ddot{\Phi} + 3H(\dot{\Psi} - 2\dot{\Phi})} \quad - (6.26)$$

$$\boxed{R_{ij} = \delta_{ij}\Big[(2a^2H^2 + a\ddot{a})(1 + 2\Phi - 2\Psi) + a^2H(6\dot{\Phi} - \dot{\Psi}) + a^2\ddot{\Phi} + k^2\Phi\Big] + k_ik_j(\Phi + \Psi)} \quad - (6.27)$$

Ricci Scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$$

$$= [-1 + 2\Psi]\left[-3\frac{\ddot{a}}{a} - \frac{k^2}{a^2}\Psi - 3\ddot{\Phi} + 3H(\dot{\Psi} - 2\dot{\Phi})\right]$$

$$+ \frac{(1-2\Phi)}{a^2}\left[3\left\{(2a^2H^2 + a\ddot{a})(1 + 2\Phi - 2\Psi) + a^2H(6\dot{\Phi} - \dot{\Psi}) + a^2\ddot{\Phi} + k^2\Phi\right\} + k^2(\Phi + \Psi)\right]$$

$$R^{(0)} = 6\left(H^2 + \frac{\ddot{a}}{a}\right)$$

$$R^{(1)} = \delta R = -6\Psi\frac{\ddot{a}}{a} + \frac{k^2}{a^2}\Psi + 3\ddot{\Phi} - 3H(\dot{\Psi} - 2\dot{\Phi}) - 6\Psi\left(2H^2 + \frac{\ddot{a}}{a}\right)$$

$$+ 3H(6\dot{\Phi} - \dot{\Psi}) + 3\ddot{\Phi} + \frac{4k^2\Phi}{a^2} + \frac{k^2\Psi}{a^2}$$

$$\boxed{\delta R = -12\Psi\left(H^2 + \frac{\ddot{a}}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\ddot{\Phi} - 6H(\dot{\Psi} - 4\dot{\Phi}) + \frac{4k^2\Phi}{a^2}}$$

→ Two components of the Einstein Eqns

↳ We solve for Φ & Ψ. $G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}$

We have 2 variables & 10 eq's. Only 2 would be useful

→ First consider time-time comp.

$$G^0{}_0 = g^{0\nu}G_{\nu 0} = g^{00}G_{00} \qquad (\because g^{0i} = 0)$$

$$G^0_0 = g^{00}\left(R_{00} - \frac{1}{2} g_{00} R\right)$$

$$= \cancel{f}\,(-1+2\Psi) R_{00} - \frac{R}{2}$$

$$= (-1+2\Psi)\left[-3\frac{\ddot a}{a} - \frac{k^2}{a^2}\Psi - 3\ddot\Phi + 3H(\dot\Psi - 2\dot\Phi)\right] - \frac{(R^{(0)} + R^{(1)})}{2}$$

$$\delta G^0_0 = -6\Psi\frac{\ddot a}{a} + \cancel{\frac{k^2}{a^2}\Psi} + \cancel{3\ddot\Phi} - 3H(\dot\Psi - 2\dot\Phi) + 6\Psi\left(H^2 + \frac{\ddot a}{a}\right) - \cancel{\frac{k^2}{a^2}\Psi} - \cancel{3\ddot\Phi}$$
$$+ 3H(\dot\Psi - 4\dot\Phi) - \frac{2k^2\Phi}{a^2}$$

$$\boxed{\delta G^0_0 = 6\Psi H^2 \cancel{+6H\dot\Phi} - 6H\dot\Phi - \frac{2k^2\Phi}{a^2}} \quad - (6.34)$$

$$T^0_0(\vec{x}, t) = -\sum_s g_s \int \frac{d^3p}{(2\pi)^3} \underbrace{E_s(p)}_{\sqrt{p^2+m_s^2}} f_s(\vec{p}, \vec{x}, t) \quad - (\text{from exercise } 3.12)$$

↳ We use expressions for f_s derived for photons, baryons, dark matter & neutrinos in ch-5.

→ For baryons & dark matter, $E_s(p) \simeq m_s \Rightarrow T^0_0 = -m_s n_s(t, \vec{x})$

$$T^0_0 = -m_s n_s(t)(1 + \delta_s(t, \vec{x}))$$

$$\boxed{T^0_0\big|_{b,c} = -\rho_s(t)(1 + \delta_s(t, \vec{x}))}$$

→ For photons eq^n (5.3) gives

$$T^0_0\big|_\gamma = -2\int \frac{d^3p}{(2\pi)^3} p\left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right]$$

$$\to -2\int \frac{d^3p}{(2\pi)^3} p f^{(0)} = -\rho_\gamma^{(0)}$$

$$\to -2\int \frac{d^3p}{(2\pi)^3} (-p^2)\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$\hookrightarrow -4\left(-2\int \frac{d^3p}{(2\pi)^3} p f^{(0)}\Theta\right)$$

$$= -4\rho_\gamma^{(0)}\Theta_0$$

{from (5.20)}

$$\boxed{T^0_0\big|_\gamma = -\rho_\gamma^{(0)}(1 + 4\Theta_0)}$$

Can also be arrived at

$$\rho_\gamma \propto T^4$$

$$\frac{\delta\rho_\gamma}{\rho_\gamma} \propto \frac{4\delta T}{T} = 4\Theta$$

→ For massless neutrinos

$$\boxed{T^0_0\big|_{\nu, m_\nu = 0} = -\rho_\nu[1 + 4\mathcal{N}_0]}$$

, the integral for neutrinos with mass can't be solved in closed form so we consider its effect later in numerical sol^n.

→ As always we neglect dark energy perturbations

Equating (6.34) to $8\pi G$ times $T^0{}_0$ & dividing by 2 we get (for first order part)

$$-3H\dot{\Phi} + 3\Psi H^2 - \frac{k^2\Phi}{a^2} = -4\pi G\left[\rho_c\delta_c + \rho_b\delta_b + 4\rho_r\Theta_0 + 4\rho_\nu \mathcal{N}_0\right]$$

↳ Fourier space & ~~coordinate~~ conformal time $(d\eta = a\,dt) \Rightarrow \left(H = \frac{a'}{a^2}\right)$

$$\boxed{k^2\Phi + 3\frac{a'}{a}\left(\Phi' - \Psi\frac{a'}{a}\right) = 4\pi G a^2\left(\rho_c\delta_c + \rho_b\delta_b + 4\rho_r\Theta_0 + 4\rho_\nu\mathcal{N}_0\right)}$$

→ In limit of no expansion ($a' = 0$) the eq^n is just a first order poisson's eq^n $\nabla^2\Phi = 4\pi G a^2 \delta\rho$

↳ first eq^n for evolution of Φ & Ψ

→ 2nd evolution eq^n using spatial part

$$G^i{}_j = g^{ik}\left[R_{kj} - \frac{g_{kj}R}{2}\right] = \frac{\delta^{ik}(1-2\Phi)R_{kj}}{a^2} - \frac{\delta^i{}_j R}{2}$$

R_{kj} ~~is~~ contains many terms proportional to δ_{kj} which on contraction with δ^{ik} gives $\delta^i{}_j$. all $\delta^i{}_j$ terms can be clubbed together

$$G^i{}_j = \underbrace{\delta^i{}_j F(\Phi,\Psi)}_{\text{all these terms contribute to trace of } G^i{}_j} + \frac{k^i k_j(\Phi+\Psi)}{a^2} \quad \text{(from 6.27)}$$

→ We consider longitudnal traceless part of $G^i{}_j$.

$$\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta^j{}_i\right)G^i{}_j = \text{First term becomes } 0 + \frac{2k^2}{3}\frac{(\Phi+\Psi)}{a^2}$$

→ We equate this with longitudnal, traceless part of $T^\mu{}_\nu$.

$$T^i{}_j(\vec{x},t) = \sum_s g_s \int \frac{d^3p}{(2\pi)^3}\frac{p^i p_j}{E_s(p)} f_s(\vec{x},\vec{p},t)$$

$$\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta^i{}_j\right)T^i{}_j(\vec{x},t) = \left(\;\cdot\;\right) p^i p_j\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta^i{}_j\right)$$

$\vec{p}\cdot\hat{k} = \mu p$ $\qquad (p^i\hat{k}_i)(p_j\hat{k}_j) = \mu^2 p^2$

$\delta^i{}_j p^i p_j = p^2$

$$\Rightarrow \left(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta^i_j\right) T^i{}_j(\vec{n}, t) = \sum_s g_s \int \frac{d^3p}{(2\pi)^3} \frac{p^2\mu^2 - p^2/3}{E_s(p)} f_s(\vec{p})$$

$$\downarrow$$

$$\mu^2 - 1/3 = \frac{2}{3} P_2(\mu)$$ (Second legendre polynomial)

picks out quadrupole moment term. → $f^{(0)}$ doesn't have quadrupole moment so it doesn't contribute & $f^{(1)}$ for dark matter & baryons only depends on position $\vec{n}$ not angle $\hat{p}$ or μ. hence for them also this integral is 0. So only contri comes from photons & neutrino

For photons $f_s(\vec{n}, \vec{p}, t) = f^{(0)}(p,t) - p\dfrac{\partial f^{(0)}}{\partial p}(p,t)\,\Theta(\vec{n}, \hat{p}, t)$

$$\text{LHS} = 2\int \frac{d^3p}{(2\pi)^3} \frac{p^2(\mu^2 - 1/3)}{p}\left[\underbrace{f^{(0)}(p,t)}_{\text{this becomes } 0} - p\frac{\partial f^{(0)}}{\partial p}(p,t)\,\Theta(\vec{n}, \hat{p}, t)\right]$$

$$= -2\int \frac{dp}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p}(p,t) \underbrace{\int_{-1}^{1} \frac{d\mu}{2} \frac{2 P_2(\mu)\Theta(\mu)}{3}}_{\frac{1}{(-i)^2}\int_{-1}^{1}\frac{d\mu}{2}\frac{2}{3}P_2(\mu)\Theta(\mu) = \Theta_2(\mu)}$$

$$= 2 \cdot \frac{2\Theta_2}{3} \int \frac{dp}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p}$$

$$= -\frac{2\Theta_2}{3} \cdot \left[2\int \frac{dp}{(2\pi^2)} 4p^3 f^{(0)}\right] = -\frac{2\Theta_2}{3} 4\rho_r = \underbrace{-\frac{8}{3}\rho_r\Theta_2}_{\text{anisotropic stress}}$$

$$\Rightarrow \frac{2}{3} \frac{k^2(\phi + \psi)}{a^2} = -8\pi G\left(-\frac{8}{3}\rho_r\Theta_2 + \rho_\nu N_2\right)$$

$$\boxed{k^2(\phi + \psi) = -32\pi G a^2(\rho_r\Theta_2 + \rho_\nu N_2)}$$

→ doesn't contribute much as during the time period where ρ_r is appreciable – (6.48) Θ_2 is negligible due to tight coupling

6.4 Tensor Perturbations → give rise to gravitational wave

Ligo → wavelength of grav. waves $\sim m$

Cosmological gw → wavelength of order ~~Mpc~~ $\sim$ Mpc

→ Tensor perturbation can be characterized by a metric perturbation.

with $h_{00} = -1$, $h_{0i} = 0$ &

$$\delta g_{ij} = a^2(t)\, h_{ij}^{TT}(t, \vec{n}) \quad , \quad h_{ij}^{TT} = \underbrace{\begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

→ perturbations assumed in x-y plane

→ choosing z-axis to be in direction of $\vec{k}$.

→ h_+ & $h_\times$ are comps of a divergenceless, traceless, symm. tensor.

Divergenceless

$\partial^i h_{ij}^{TT} \stackrel{?}{=} 0$

$i k^i h_{ij}^{TT} \stackrel{?}{=} 0$

since $\hat{k} = \hat{e}_z \Rightarrow k^x = k^y = 0$
$k^z = k$

$i k h_{zj}^{TT} \stackrel{?}{=} 0$

$\boxed{0 = 0}$

6.4.1 Christoffel Symbol for Tensor Perturbations

(i) $\Gamma^0_{00} = \frac{1}{2} g^{0\alpha}\left[\partial_0 g_{0\alpha} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}\right] = 0$

(ii) $\Gamma^0_{i0} = \frac{1}{2} g^{0\alpha}\left[\partial_i g_{0\alpha} + \partial_0 g_{\alpha i} - \partial_\alpha g_{0i}\right] = 0$

(iii) $\Gamma^0_{ij} = \frac{1}{2} g^{0\alpha}\left[\partial_i g_{j\alpha} + \partial_j g_{i\alpha} - \partial_\alpha g_{ij}\right] = \frac{1}{2}\partial_0 g_{ij} = H g_{ij} + \frac{a^2}{2}\partial_0 h_{ij}^{TT}$

(iv) $\Gamma^i_{00} = \frac{1}{2} g^{i\alpha}\left[\partial_0 g_{\alpha 0} + \partial_0 g_{0\alpha} - \partial_\alpha g_{00}\right] = 0$

(v) $\Gamma^i_{0j} = \frac{1}{2} g^{i\alpha}\left[\partial_0 g_{j\alpha} + \partial_j g_{\alpha 0} - \partial_\alpha g_{0j}\right] = \frac{g^{ik}}{2}\partial_0 g_{jk}$

$\boxed{\Gamma^i_{0j} = H\delta^i_j + \frac{1}{2}\partial_0 h_{ij}^{TT}} \quad \Leftarrow \; = \frac{g^{ik}}{2}\left[2H g_{jk} + a^2\partial_0 h_{jk}^{TT}\right]$

$g^{ik} = (g^{ik(0)} - h^{ik}) \qquad g^{ik} a^2 \partial_0 h_{jk}^{TT} = \frac{a^2 \partial_0 h^{ik}}{a^2}$

$$\Gamma^{i}_{jk} = \frac{1}{2} g^{i\alpha}\left[\partial_j g_{k\alpha} + \partial_k g_{\alpha j} - \partial_\alpha g_{jk}\right] \quad (\alpha \text{ can't be } 0)$$

$$= \frac{1}{2}\frac{1}{a^2}\delta^{i\alpha}\left[\cancel{\partial_j (a^2}(a^2)\left[\partial_j h^{TT}_{k\alpha} + \partial_k h^{TT}_{j\alpha} - \partial_\alpha h^{TT}_{jk}\right]\right.$$

$$\boxed{\Gamma^{i}_{jk} = \frac{i}{2}\left[k_k h^{TT}_{ij} + k_j h^{TT}_{ik} - k_i h^{TT}_{jk}\right]}$$

6.4.2 Ricci tensor for tensor perturbations

$$R_{00} = \cancel{\partial_\alpha \Gamma^{\alpha}_{00}} - \partial_0 \Gamma^{\alpha}_{\alpha 0} + \cancel{\Gamma^{\alpha}_{\alpha k}\Gamma^{k}_{00}} - \Gamma^{\alpha}_{0k}\Gamma^{k}_{\alpha 0}$$

$$= -3\partial_0 H - \cancel{\frac{\text{trace}(h^{TT}_{ij})}{2}} - 3H^2 - \cancel{\sum_i H\partial_0 h^{TT}_{ii}}$$

$$R^{(1)}_{00} = 0$$

$$R_{ij} = \underbrace{\partial_\alpha \Gamma^{\alpha}_{ij} - \partial_j \Gamma^{\alpha}_{i\alpha}} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ij} - \Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{i\alpha}$$

Consider $\partial_\alpha \Gamma^{\alpha}_{ij} - \partial_j \Gamma^{\alpha}_{i\alpha} = \underbrace{\partial_0 \Gamma^{0}_{ij}}_{\frac{\ddot{g}_{ij}}{2}} + \partial_k \Gamma^{k}_{ij} - \partial_j \underbrace{\Gamma^{k}_{ik}}_{\frac{i}{2}[k_i h^{TT}_{kk}]}$

summed over k $= 0$

$$\partial_\alpha \Gamma^{\alpha}_{ij} - \partial_j \Gamma^{\alpha}_{i\alpha} = \frac{\ddot{g}_{ij}}{2} + \left[\frac{1}{2}\right]\left[-\underbrace{k_k k_i h_{jk}} - \underbrace{k_j k_k h^{TT}_{ik}} + k^2 h^{TT}_{ij}\right]$$

Vanish due to transverse nature of metric

$$\boxed{\partial_\alpha \Gamma^{\alpha}_{ij} - \partial_j \Gamma^{\alpha}_{i\alpha} = \frac{\ddot{g}_{ij}}{2} + \frac{k^2 h^{TT}_{ij}}{2}}$$

→ 3rd term $\Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ij}$ is non zero only when $\alpha \neq 0$

$$\Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ij} = \Gamma^{k}_{k0}\Gamma^{0}_{ij} + \underbrace{\Gamma^{k}_{kl}}_{\sim h}\underbrace{\Gamma^{l}_{ij}}_{\sim h} \quad (2^{nd} \text{ order})$$

$$= \left(H\delta^{k}_{k} + \frac{1}{2}\partial_0 h^{TT}_{kk}\right)\left(Hg_{ij} + \frac{a^2}{2}\partial_0 h^{TT}_{ij}\right) = \frac{3}{2}\partial_0 g_{ij}$$

$$\Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{i\alpha} = \Gamma^{0}_{kj}\Gamma^{k}_{i0} + \Gamma^{l}_{0j}\Gamma^{0}_{il} + \underbrace{\Gamma^{m}_{nj}\Gamma^{n}_{im}}_{\sim h^2}$$

$$= \cancel{\frac{1}{2}(\partial_0 g_{kj})}\,\frac{a}{a}\left(H g_{kj} + \frac{a^2}{2}\partial_0 h^{TT}_{kj}\right)\left(H\delta^k_i + \frac{1}{2}\partial_0 h^{TT}_{ki}\right)$$

$$+ \Gamma^{l}_{0j}\Gamma^{0}_{il}$$

$$= \frac{a}{a}\left(H^2 g_{ij} + \frac{a^2}{2}H\,\partial_0 h^{TT}_{ij} + \frac{H}{2}\underbrace{g_{kj}}_{\substack{\text{zeroth}\\ (\delta^k_j a^2)}}\partial_0 h^{TT}_{ki}\right) + \Gamma^{l}_{0j}\Gamma^{0}_{il}$$

$$= H^2 g_{ij} + \frac{a^2}{2}H\partial_0 h^{TT}_{ij} + \frac{Ha^2}{2}\partial_0 h^{TT}_{ij} + (i\leftrightarrow j)$$

$$\boxed{\Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{i\alpha} = 2H^2 g_{ij} + 2a^2 H\,\partial_0 h^{TT}_{ij}}$$

$$R_{ij} = \frac{\ddot{g}_{ij}}{2} + \frac{k^2}{2}h^{TT}_{ij} + \frac{3}{2}H\dot{g}_{ij} - 2H^2 g_{ij} - 2a^2 H\,\dot{h}^{TT}_{ij}$$

$$\dot{g}_{ij} = 2Hg_{ij} + a^2\dot{h}^{TT}_{ij}$$

$$\ddot{g}_{ij} = 2g_{ij}\left(\frac{\ddot{a}}{a} - H^2\right) + 2H\left(2Hg_{ij} + a^2\dot{h}^{TT}_{ij}\right) + 2a^2 H\dot{h}^{TT}_{ij} + a^2\ddot{h}^{TT}_{ij}$$

$$= 2g_{ij}\left(\frac{\ddot{a}}{a} + H^2\right) + 4a^2 H\dot{h}^{TT}_{ij} + a^2\ddot{h}^{TT}_{ij}$$

$$R_{ij} = g_{ij}\left(\frac{\ddot{a}}{a} + H^2\right) + 2a^2 H\dot{h}^{TT}_{ij} + \frac{a^2}{2}\ddot{h}^{TT}_{ij} + \frac{3}{2}H\left(2Hg_{ij} + a^2\dot{h}^{TT}_{ij}\right)$$

$$- 2H^2 g_{ij} - 2a^2 H\dot{g}^{TT}_{ij}$$

$$\boxed{R_{ij} = g_{ij}\left(\frac{\ddot{a}}{a} + 2H^2\right) + \frac{3}{2}a^2 H\dot{h}^{TT}_{ij} + \frac{a^2}{2}\ddot{h}^{TT}_{ij} + \frac{k^2}{2}h^{TT}_{ij}}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij}$$

→ We first prove the fact that the Ricci scalar doesn't contain a contri from h_{ij}^{TT} at first order.

Only produce zero order term

$$R = \underbrace{g^{00} R_{00}}_{\text{zeroth order}} + g^{ij} R_{ij} = \frac{1}{a^2}(\delta_{ij} - h_{ij}^{TT}) \left[(\delta_{ij} + h_{ij}^{TT}) \left(\frac{\ddot{a}}{a} + 2H^2 \right) + h_{ij}^{TT}(\quad) \right]$$

$$= \frac{1}{a^2} \underbrace{\delta_{ij} h_{ij}^{TT}}_{0} (\quad) = 0 \quad (\text{Traceless } h_{ij}^{TT})$$

6.4.3 Einstein's equations for tensor perturbations

$$\delta G^i{}_j = \delta\left(R^i{}_j + \frac{1}{2} g^i{}_j R\right) = \delta R^i{}_j + \frac{R}{2} \delta g^i{}_j$$

doesn't contain any pert. term

$$G^i{}_j = g^{ik}\left(R_{kj} - \frac{g_{kj} R}{2}\right) = R^i{}_j - \frac{\delta^i{}_j R}{2}$$

$$\delta G^i{}_j = \delta R^i{}_j - \frac{\delta^i{}_j \underbrace{(\delta R)}_{=0}}{2}$$

$$\delta G^i{}_j = \delta R^i{}_j = \delta\left[\underbrace{g^{ik}}_{\frac{1}{a^2}(\delta^{ik} - h^{TT}_{ik})} \left(g_{kj}\left(\frac{\ddot{a}}{a} + 2H^2\right) + \frac{3}{2} a^2 H \dot{h}_{kj}^{TT} + \frac{a^2}{2} \ddot{h}_{kj}^{TT} + \frac{k^2}{2} h_{kj}^{TT} \right)\right]$$

Here δ implies selection of first order term

$$= \delta^{ik}\left[\frac{3}{2} H \dot{h}_{kj}^{TT} + \frac{1}{2} \ddot{h}_{kj}^{TT} + \frac{k^2}{2a^2} h_{kj}^{TT} \right]$$

→ $\delta R^i{}_j \propto h_{ij}^{TT}$ & derivatives → $\delta G^1{}_1 = -\delta G^2{}_2$ ($\because h_{11}^{TT} = -h_{22}^{TT} = h_+$)

→ $\delta G^1{}_1 - \delta G^2{}_2 = 3H\dot{h}_+ + \ddot{h}_+ + \frac{k^2}{a^2} h_+$

→ Considering conformal time $\dot{h}_+ = \frac{h_+'}{a}$, $\ddot{h}_+ = \frac{d}{dt}\left(\frac{h_+'}{a}\right) = \frac{h_+''}{a^2} - \frac{h_+' a'}{a^3}$

$$\Rightarrow \delta G^1{}_1 - \delta G^2{}_2 = \frac{h_+''}{a^2} + 2h_+' \frac{a'}{a^3} + \frac{k^2 h_+}{a^2} \Rightarrow a^2[\delta G^1{}_1 - \delta G^2{}_2] = h_+'' + \frac{2a'}{a} h_+' + k^2 h_+$$

Note for $T^1{}_1 - T^2{}_2 = -\sum_s g_s \int \frac{d^3p}{(2\pi)^3} \frac{\left(p^1 p_1 f_s(\vec{n}, \vec{p}, t) - p p^2_2 f_s(\vec{n}, \vec{p}, t)\right)}{E_s(p)}$

→ first consider CDM & baryons

$\rightarrow$ Consider $\delta G'_2 = \delta R'_2 = \delta^{ik}\left[\frac{3}{2}H\dot{h}^{TT}_{kj} + \frac{\ddot{h}^{TT}_{kj}}{2} + \frac{k^2}{2a^2}h^{TT}_{kj}\right]$

$(i=1, j=2)$

$= \frac{3}{2}H\dot{h}^{TT}_{12} + \frac{\ddot{h}^{TT}_{12}}{2} + \frac{k^2}{2a}h^{TT}_{12}$

$= \frac{1}{2}\left(h''_{12} + 2\frac{a'}{a}h'_{12} + k^2 h_{12}\right)$; $h_{12} = h_\times$

Note that $T'_2 = \int \frac{p' p_2}{(\ )}\left(p f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\right) = \int \frac{d^3p}{(2\pi)^3}\, p^2 \sin^2\theta \sin\phi\cos\phi\,(\ )$

due to this $\underline{T'_2 \text{ vanish}}$

damping term

$\Rightarrow \boxed{h''_+ + \underbrace{2\frac{a'}{a}h'_+}_{} + k^2 h_+ = 0}$; $+ = +, \times$

$\hookrightarrow$ wave eqⁿ (check by subs. $a' = 0$)

$\hookrightarrow$ Neglecting damping term the two solⁿs we get $h_+ \propto e^{\pm ik\eta}$

This implies

$h_+(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\left[A(\vec{k})e^{ik\eta} + B(\vec{k})e^{-ik\eta}\right]$.

in real space

$\rightarrow$ We consider these eqⁿ for a purely matter & radiation dominated universe. (Eqn - 6.12)

(i) Radiation dominated

$H = \frac{\dot{a}}{a} = H_0\left[\Omega_{R_0}(1+z)^4\right]^{1/2} = \frac{H_0\Omega_{R_0}^{1/2}}{a^2} = \frac{a'}{a^2} \Rightarrow a' = H_0\Omega_{R_0}^{1/2}$

$a = H_0\Omega_{R_0}^{1/2}\eta$

$\Rightarrow \boxed{\frac{a'}{a} = \frac{1}{\eta}}$

Subs. in eqⁿ (6.73)

$h''_+ + \cancel{2a}\,\frac{2}{\eta}h'_+ + k^2 h_+ = 0$

Solⁿ $\rightarrow h_+ = \frac{Ae^{ik\eta}}{k\eta} + \frac{Be^{-ik\eta}}{k\eta}$ $\rightarrow$ (as $\eta \uparrow$ amplitude decreases)

$\sim \left(\frac{\cos k\eta}{k\eta}, \frac{\sin k\eta}{k\eta}\right)$

(ii) Matter dominated

$$\frac{a'}{a} = \frac{2}{\eta} \qquad \text{Eq}^n \rightarrow h_+'' + \frac{4}{\eta} h_+' + k^2 h_+ = 0$$

Sol^{n}s $\rightarrow$ $$\frac{\frac{\sin k\eta}{k\eta} - \cos k\eta}{\eta^{3/2}\sqrt{k\eta}} \,, \quad \frac{+\sin k\eta + \frac{\cos k\eta}{k\eta}}{\eta^{3/2}\sqrt{k\eta}} \qquad (n \rightarrow \eta)$$

Nice discussion after fig 6.2