Ch-8 Growth of Structure: Linear Theory

→ Nice intro in Dodelson

8.1 Prelude

→ Gravitational instability is responsible for structure formation. With time matter accumulates in slightly overdense regions. Initially very small overdensities (10^{-4}) grew to form very significant structures.

→ This clumping is slowed down by (i) expansion of universe (ii) pressure perturbation in baryons & photons which increase in proportion to density. As gas clumped region pressure is higher & since gas flows from high to low pressure regions, the clumping is slowed down.

8.1.1 Three stages of evolution

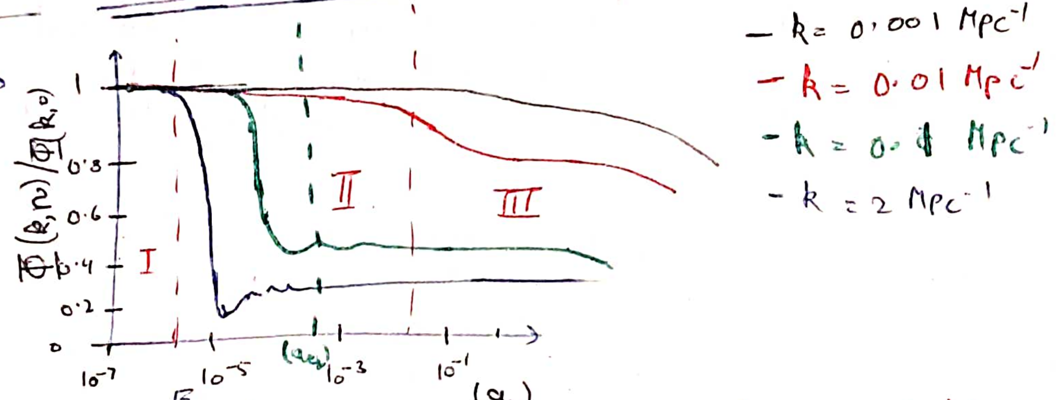

Fig 8.1

→ From fig 8.1 we can roughly divide evolution of Φ into three stages

(i) Initially all modes are out of horizon ($k\eta \ll 1$) & Φ is const. (Just after inflation)

(ii) At intermediate times, wavelength enter horizon & universe evolve from radiation ($a \ll a_{eq}$) to matter dominated era ($a \gg a_{eq}$). Here we observe that large scale modes which enter horizon well after a_{eq} evolve much differently than small scale modes.

(iii) At late times all modes evolve identically again, remaining constant during matter domination, before decaying once dark energy become important

→ We can relate the potential at late times (which we are able to observe) to the primordial curvature perturbation $\mathcal{R}$ (2) during schematically:

$$\Phi(\vec{k}, a) = \underbrace{\overset{\text{explained later}}{\frac{3}{5}\mathcal{R}(\vec{k})}}_{\text{In phase I}} \times \underbrace{\{\text{Transfer Function}(k)\}}_{\text{In phase II}} \times \underbrace{\{\text{Growth factor}(a)\}}_{\substack{\text{In phase III} \\ \text{(Evolution k independent)}}}$$

→ Note that even the largest wavelength perturbations decline slightly as the universe passes through epoch of equality. This decline is conventionally removed so that the transfer function on large scales is equal to 1.

→ $$\boxed{T(k) = \frac{\Phi(\vec{k}, a_{late})}{\Phi_{large\text{-}scale}(\vec{k}, a_{late})}}$$

a_{late} → epoch deep in matter domination

$\Phi_{large\text{-}scale}$ → (Primordial Φ decreased by a small amount, strictly it is the sol^n of the gravitational potential for modes that entered the horizon well in the matter-dominated epoch).

→ Growth factor

$$\frac{\Phi(\vec{k}, a)}{\Phi(\vec{k}, a_{late})} \equiv \frac{\overset{\text{growth factor}}{D_+(a)}}{a} \qquad (a > a_{late})$$

→ During matter-domination potential is constant $D_+(a) = a$. With these conventions, we have

$$\boxed{\Phi(\vec{k}, a) = \frac{3}{5}\mathcal{R}(\vec{k})\, T(k)\, \frac{D_+(a)}{a}} \qquad (a > a_{late}) \qquad -(8.4)$$

→ CDM overdensity δ_c evolves with Φ as shown in 8.2. While Φ remains constant in the third phase (all modes are within the horizon) the overdensity grows in time. Hence the name "growth"-factor for $D_+(a)$.

→ In late universe, baryons closely follow dark matter (as baryons have already decoupled from radiation) so they can be described together in form of the total matter over density δ_m.

→ We try to find power spectrum of matter distribution in terms of primordial power spectrum.

We use eq^n (6.80) to relate potential to overdensity term in large-k, no-radiation limit.

$$k^2 \bar{\Phi} = 4\pi G a^2 \bar{\rho}_m \delta_m$$

$$k^2 \Phi(\vec{k},a) = 4\pi G \bar{\rho}_m(a) a^2 \delta_m(\vec{k},a) \quad - (8.5) \quad (a > a_{late},\ k >> aH)$$

↳ eq^n applicable only for modes $k >> aH$. But since most of our precise measurements (& observations) are for modes that satisfy $k >> aH$ (as aH is so small in current epochs) so this works for us.

Using $\frac{\rho_m}{\rho_{cr}} = \frac{\Omega_{m0}}{a^3}$ & $\frac{8\pi G}{3H_0^2} = \frac{1}{\rho_{cr}}$ $\rightarrow 4\pi G \rho_{cr} = \frac{3H_0^2}{2}$

$$\frac{3H_0^2}{2} \frac{\Omega_m}{a} \delta_m(\vec{k},a) = k^2 \Phi(\vec{k},a)$$

$$\boxed{\delta_m(\vec{k},a) = \frac{2k^2 a\, \Phi(\vec{k},a)}{3\Omega_m H_0^2}} \quad - (8.6)$$

Using (8.4)

$$\boxed{\delta_m(\vec{k},a) = \frac{2}{5}\frac{k^2}{\Omega_m H_0^2} \mathcal{R}(\vec{k})\, T(k)\, D_+(a)} \quad (8.7) \quad (a >> a_{late},\ k >> aH)$$

This eq^n holds regardless of how initial perturbation $\mathcal{R}(\vec{k})$ was generated (as long as it is adiabatic).

From eq^n (7.95) we have $P_{\mathcal{R}}(k) = \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_p}\right)^{n_s - 1}$

Linear Power spectrum of matter at late times

$$P_L(k,a) = \left[\frac{2}{5}\frac{k^2\, T(k) D_+(a)}{\Omega_m H_0^2}\right]^2 P_{\mathcal{R}}(k)$$

$$\boxed{P_L(k,a) = \frac{8\pi^2}{25}\frac{A_s}{\Omega_m^2} D_+^2(a)\, T^2(k)\, \frac{k^{n_s}}{H_0^4\, k_p^{n_s-1}}} \quad - (8.8)$$

↳ Dimensions (L^{+3})

on large scales where $T(k) = 1$

$\Rightarrow P_L(k) \propto k^{n_s}$

→ Fig 8.3 ~~gives the~~ shows the matter power spectrum for our ΛCDM cosmology. Some important conclusions can be drawn.

→ On large scales (smaller k values) ~~($k << aH$)~~ from eq 8.8 we have $P_L(k) \propto k$, which is shown in the figure

(4)

→ On small scales the power spectrum turns over. The small scale mode enters the horizon well before matter-radiation equality & hence have $T(k) << 1$ (as can be seen decaying potential during radiation regime). The effect of small $T(k)$ can be seen on matter perturbations from figure 8.2. During $a \sim 10^{-5}$ - $a \sim 10^{-4}$ growth of δ is retarded. The turnover in the power spectrum would be at a scale k_{eq} which enters the horizon at matter/radiation epoch.

Measuring this scale allows us to constrain the amount of matter in universe. (Fig 8.8)

→ Discussion about k_{NL}

→ Scale k_{NL} above which non linearities cannot be ignored.

→ To estimate this, we use the variance of (linear) density perturbations generated by modes with a logarithmic wavenumber $d \ln k$ centered around k

$$\Delta_L^2(k, a) = \frac{1}{\epsilon} \int_{|\ln k' - \ln k| < \epsilon} \frac{d^3 k'}{(2\pi)^3} P_L(k', a)$$

$$= \frac{1}{\epsilon} \int \frac{(k')^2 dk'}{(2\pi)^3} \int d\Omega' \, P_L(k', a)$$

$$= \frac{1}{\epsilon} \int (k')^2 dk' \, P_L(k', a) \frac{4\pi}{(2\pi)^3} = \frac{\cancel{P_L(k, a)}}{2\pi^2 \epsilon} \int P_L(k', a) \, dk' (k')^2$$

since dk is very small we can write

$$\Delta_L^2(k, a) = \frac{P_L(k, a) k^2}{2\pi^2 \epsilon} \int_{k e^{-\epsilon}}^{k e^{\epsilon}} dk' = \frac{P_L(k, a) k^2}{2\pi^2 \epsilon} \left(k e^{\epsilon} - k e^{-\epsilon} \right)$$

$$\boxed{\Delta_L^2(k, a) = \frac{P_L(k, a) \, k^3}{\pi^2 \epsilon}} \quad - (D)$$

→ $\Delta_L^2 << 1$ → small inhomogenities, $\Delta_L^2 \gtrsim 1$ non linear perturbations

(D) → At earlier times the structure was not as evolved so the non linear scale was small & hence k_{NL} was large.

→ The power (linear) spectrum show in Fig 8.3 is valid only for $k \leq k_{NL}$.

8.1.2 Closing the Boltzmann Hierarchy

(5)

→ We try to ~~deteuime~~ determine the evolution equations for dark matter overdensity δ_c.

→ Ideally we should consider all eq^n from chapter 5 & 6. But we don't need complete set of eq^{n}s ~~as~~ bcoz before recomb^n only Θ_0, Θ_1 are enough to characterize photons (as they are tightly coupled to electron/proton plasma. After the ~~recombination~~ decoupling, for matter distributions the photons become irrelevant (after a_*). (Matter dominated era, potential dominated by dark matter itself)

→ We'll also neglect higher moments for neutrinos, though it is incorrect.

→ We arrive at eq's 8.10 – 8.13 in exercise-1

$$\Theta_{r,0}' + k\Theta_{r,1} = -\Phi' \quad -(8.10)$$

$$\Theta_{r,1}' - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \quad -(8.11)$$

$$\delta_c' + ikv_c = -3\Phi'$$

$$v_c' + \frac{a'}{a}v_c = ik\Phi$$

→ We need one more eq^n for Φ.

→ One more eq^n can be either (6.41)

$$k^2\Phi + 3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Phi\right) = 4\pi G a^2\left[\rho_c\delta_c + 4\rho_r\Theta_{r,0}\right] \quad -(8.14)$$

or

$$k^2\Phi = 4\pi G a^2\left[\rho_c\delta_c + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_c v_c + 4\rho_r\Theta_{r,1}\right)\right]$$

→ In section 8.2 we'll try to solve these eq^n analytically.

There are three regimes

(i) Super-horizon regime	(ii) Horizon entry	(iii) Sub-horizon evolution
$k\eta << 1$	η has inc. sufficiently, $k\eta > 1$	$k\eta >> 1$

8.2 Large Scales

→ In case of large scale modes, horizon crossing ($k\eta = 1$) occours after matter-radiation transition. So we first obtain the super-horizon solution which is valid in through matter-radiation epoch.

8.2.1 Super Horizon Solution

→ For modes far outside horizoon $k\eta \ll 1$, we can drop all terms in the evolution eq^n that depend on k. We get

$$\Theta_{r,0}' = -\Phi' \qquad - 8.16$$

$$\delta_c' = -3\Phi' \qquad - 8.17$$

$$3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Phi\right) = 4\pi G a^2 \left[\rho_c \delta_c + 4\rho_r \Theta_{r,0}\right] \qquad 8.18$$

(from 8.14)

From 8.16 & 8.17 we have $\left[\delta_c' = 3\Theta_{r,0}' \Rightarrow \delta_c = 3\Theta_{r,0}\right]$ (Also from adiabatic perturb.)

$$3\frac{a'}{a}\left[\Phi' + \frac{a'}{a}\Phi\right] = 4\pi G a^2 \left[\rho_c \delta_c + 4\rho_r \frac{\delta_c}{3}\right]$$

Now we introduce a variable

$$\boxed{y = \frac{\rho_m}{\rho_r} = \frac{\cancel{\rho_m(a=a_{eq})}\, a_{eq}^3\, a^4}{a^3\, \cancel{\rho_r(a=a_{eq})}\, a_{eq}^4} = \frac{a}{a_{eq}}}$$

Since we are ignoring Baryons (this actually make analytical solution more correct) we can even write

$$y = \frac{\rho_c}{\rho_r}$$

$$\Rightarrow 3\frac{a'}{a}\left[\Phi' + \frac{a'}{a}\Phi\right] = 4\pi G a^2 \rho_c \delta_c \left[1 + \frac{4}{3y}\right] \qquad - 8.19$$

→ Eq^n 8.17 & 8.19 would serve as two first order eq^n's. We try to combine them & make a second order one.

First we see that

$$\frac{d}{d\eta} = \frac{d}{dy}\frac{dy}{d\eta}\frac{d}{dy} = \frac{d}{d\eta}\left(\frac{a}{a_{eq}}\right)\frac{d}{dy}$$

$$\frac{d}{d\eta} = \frac{a'}{a_{eq}}\frac{d}{dy} = aHy\frac{d}{dy} \qquad -(8.21)$$

8.19 becomes

$$3\frac{a'}{a}\left[aHy\frac{d\Phi}{dy}+\frac{\cancel{a}}{a}aH\Phi\right]=\underbrace{4\pi G a^2 \mathcal{L}}\,\delta_c\left[1+\frac{4}{3y}\right]$$

$$\cancel{3a'H\left[y\,d\right.}$$

$$\frac{8\pi G}{3}\rho=H^2=\frac{8\pi G}{3}(\rho_c+\rho_r)$$

$$=\frac{8\pi G}{3}\rho_c\left(1+\frac{1}{y}\right)$$

$$\cancel{3}\cancel{a'}\cancel{H}\left[y\frac{d\Phi}{dy}+\Phi\right]=\frac{\cancel{3H^2}\cancel{a^2y}}{2\cancel{y}}\frac{y}{y+1}\delta_c\left[1+\frac{4}{3y}\right]$$

$$y\frac{d\Phi}{dy}+\Phi=\delta_c\left[1+\frac{4}{3y}\right]\cdot\frac{y}{y+1}=\frac{3y+4}{6(y+1)}\delta_c$$

$$\delta_c=\frac{6(y+1)}{3y+4}\left(y\frac{d\Phi}{dy}+\Phi\right)$$

Subs into 8.17 we have

$$-3\frac{d\Phi}{dy}=\frac{d}{dy}\left[\frac{6(y+1)}{3y+4}\left(y\frac{d\Phi}{dy}+\Phi\right)\right]$$

$$-3\frac{d\Phi}{dy}=\left(y\frac{d\Phi}{dy}+\Phi\right)\left(\frac{18y+24-18y-18}{(3y+4)^2}\right)$$

$$+\frac{6(y+1)}{3y+4}\left(y\frac{d^2\Phi}{dy^2}+2\frac{d\Phi}{dy}\right)$$

$$\frac{6(y+1)y}{3y+4}\frac{d^2\Phi}{dy^2}+\frac{d\Phi}{dy}\underbrace{\left[\frac{6y}{(3y+4)^2}+\frac{12(y+1)}{3y+4}+3\right]}_{\frac{6y+36y^2+84y+48+27y^2+48+72y}{(3y+4)^2}}+\Phi\left[\frac{6}{(3y+4)^2}\right]=0$$

$$\frac{d^2\Phi}{dy^2}+\frac{d\Phi}{dy}\left[\frac{21y^2+54y+32}{2y(y+1)(3y+4)}\right]+\frac{\Phi}{y(y+1)(3y+4)}=0 \quad (8.24)$$

$$\rightarrow \quad U = \frac{y^3}{\sqrt{1+y}}\,\Phi$$

$$\frac{du}{dy} = \frac{d\Phi}{dy}\frac{y^3}{\sqrt{1+y}} + \Phi\,\frac{\sqrt{1+y}\,3y^2 - \frac{y^3}{2\sqrt{1+y}}}{(1+y)} = \frac{d\Phi}{dy}\frac{y^3}{\sqrt{1+y}} + \Phi\,\frac{(6+5y)}{2(1+y)^{3/2}}$$

$$\frac{d^2u}{dy^2} = \frac{d^2\Phi}{dy^2}\frac{y^3}{\sqrt{1+y}} + 2\frac{d\Phi}{dy}\cdot\frac{6+5y}{2(1+y)^{3/2}} + \Phi\,\frac{(1+y)^{3/2}\cdot 5 - (5y+6)\times\frac{3}{2}\sqrt{(1+y)}}{2\,(1+y)^3}$$

$$\frac{d^2u}{dy^2} = \frac{d^2\Phi}{dy^2}\frac{y^3}{\sqrt{1+y}} + 2\frac{d\Phi}{dy}\,\frac{6+5y}{2(1+y)^{3/2}} + \Phi\,\frac{(-5y-8)}{4(1+y)^{5/2}}$$

It can be shown that 8.24 becomes

$$\frac{d^2u}{dy^2} + \frac{du}{dy}\left[-\frac{2}{y} + \frac{3/2}{1+y} - \frac{3}{3y+4}\right] = 0$$

$$\Rightarrow \frac{du'}{u'} = dy\left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4}\right] \qquad \left(u' = \frac{du}{dy}\right)$$

$$\Rightarrow \ln u' = 2\ln y - \frac{3}{2}\ln(1+y) + \ln(3y+4) + c$$

$$U' = A\,\frac{y^2(3y+4)}{(1+y)^{3/2}}$$

$$\frac{y^3\Phi}{\sqrt{1+y}} = A\int_0^y d\tilde{y}\,\frac{(\tilde{y}^2)\cdot(3\tilde{y}+4)}{(1+\tilde{y})^{3/2}}$$

Substitute $\tilde{n} \equiv \sqrt{1+\tilde{y}}$

$$n^2 = 1+\tilde{y} \qquad \tilde{y} = \tilde{n}^2 - 1$$

$$2\tilde{n}\,d\tilde{n} = d\tilde{y}$$

$$\frac{y^3\Phi}{\sqrt{1+y}} = A\int_0^{n^2-1} 2\tilde{n}\,d\tilde{n}\,\frac{(\tilde{n}^2-1)^2(3\tilde{n}^2+1)}{(\tilde{n})^3} = 2A\int_0^{n^2-1} d\tilde{n}\,\frac{[\tilde{n}^4 - 2\tilde{n}^2+1][3\tilde{n}^2+1]}{(\tilde{n})^2}$$

$$= 2A\int_0^{n^2-1} d\tilde{n}\,\frac{[3\tilde{n}^6 - 5\tilde{n}^4 + \tilde{n}^2 + 1]}{(\tilde{n})^2}$$

(9)

$$\rightarrow \cancel{\frac{y^3\Phi}{\sqrt{1+y}} = 2A\left[\frac{3}{5}(1+y)^{5/2} - \frac{5}{3}(1+y)^{3/2} + \sqrt{1+y} - \frac{2}{(1+y)^{3/2}}\right]}$$

$$\cancel{= 2A} \Big[$$

$$\frac{y^3\Phi}{\sqrt{1+y}} = 2A\int_0^y \frac{d\tilde{y}}{2\sqrt{1+\tilde{y}}}\left[3(1+\tilde{y})^2 - 5(1+\tilde{y}) + 1 + \frac{1}{(1+\tilde{y})}\right]$$

$$= 2A\left[\frac{3}{2}\times\frac{2}{5}\left[(1+y)^{5/2}-1\right] - \frac{5\times 2}{2\times 3}\left((1+y)^{3/2}-1\right) + \frac{1}{2\times 1}\times 2\left((1+y)^{1/2}-1\right) + \frac{1}{2\times(-1/2)}\left(\frac{1}{\sqrt{1+y}}-1\right)\right]$$

$$\frac{y^3\Phi}{\cancel{\sqrt{1+y}}} = 2A\left[\frac{3}{5}(1+y)^3 - \frac{5}{3}(1+y)^2 + (1+y) - 1 + \frac{16}{15}\sqrt{1+y}\right]$$

$$y^3\Phi = 2A\left[\frac{3y^3}{5} + \frac{9y^2}{5} - \frac{5y^2}{3} + \frac{9y}{5} - \frac{10y}{3} + y + \frac{3}{5} - \frac{5}{3} + \frac{16}{15}\sqrt{1+y}\right]$$

$$y^3\Phi = 2A\left[\frac{3y^3}{5} + \frac{2y^2}{15} + - \frac{8y}{15} - \frac{16}{15} + \frac{16}{15}\sqrt{1+y}\right]$$

$$\Phi = \frac{2A}{y^3\times 15}\left[9y^3 + 2y^2 - 8y - \cancel{16} \; 16 + 16\sqrt{1+y}\right]$$

at small y eqn 8.30 becomes

$\rightarrow$ A can be determined ~~as a~~,

$$\frac{y^3\Phi}{\sqrt{1+y}} = A\int_0^y d\tilde{y}\, 4\tilde{y}^2 = \frac{4A}{3}y^3 \Rightarrow \boxed{\Phi(0) = \frac{4A}{3}}$$

$$\Rightarrow \boxed{\Phi = \frac{1}{10y^3}\left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right]\Phi(\vec{k},0)} \quad - (8.31)$$

expression for potential on super horizon scale.

$$\left(y = \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}\right)$$

→ At small y (small a) this exp. sets $\Phi(k,y) = \Phi(k,0)$.

→ At large y ($a \gg a_{eq}$) (matter dominated era), y^3 term starts dominating. $\Phi \to \frac{9}{10}\Phi(0)$. Potential on even the largest scale drops by $\frac{9}{10}$ as the universe passes through epoch of equality

→ We obtain a relation b/w Super-Horizon grav. potential Φ & curvature perturbation $\mathcal{R}$. From sect 7.5 we have $\Phi \simeq -\Psi = \frac{2\mathcal{R}}{3}$ during Radiation domination. Since $\mathcal{R}$ is always conserved outside horizon we have

$$\Phi(k,\eta)\Big|_{\text{super-horizon}} = \begin{cases} \frac{2}{3}\mathcal{R}(\vec{k}) & \text{Radiation domination} \\ \frac{3}{5}\mathcal{R}(\vec{k}) & \text{Matter domination} \end{cases}$$

→ Note that the solution $\Phi(k,\eta) = \frac{3}{5}\mathcal{R}(\vec{k})$ is valid even inside horizon as can be seen from fig. 1.
↳ (Proof in Sec. 8.2.2)

→ Fig 8.6 compares this analytic solution with the actual numerical sol^n.

8.2.2 Through Horizon Crossing

→ For the mode $k = \cancel{10^{-3}}\ 10^{-3}\,h\ \text{Mpc}^{-1}$, the mode enters the horizon at $\eta \sim k^{-1} = 1000\ h^{-1}\ \text{Mpc}$ which corresponds to $a \sim 0.006$. The potential remains constant as the mode crosses the horizon. This result is valid for matter dominated universe. We'll prove this now.

→ We use the set of five eqns (8.10 – 8.14)(8.15) in the limit that radiation is negligible. This would allow us to neglect eq^n (8.10 – 8.11). We're left with 8.12 & 8.13 and then we choose 8.15 since it's an algebraic eq^n. This would give us two first order eq^n in Φ having two general sol^n.

→ The initial conditions for our equations is that $\Phi' = 0$ (since superhorizon solutions in deep matter epochs is constant Φ). If one of the two general sol^n is Φ = const then it is the sol^n.

→ So we check if Φ = const. is actually a sol^n.

$$\delta_c' + iku_c = \Phi' \qquad 8.33$$

$$u_c' + aHu_c = ik\Phi \qquad 8.34$$

$$k^2\Phi = \frac{3a^2H^2}{2}\left[\delta_c + \frac{3aH\, iu_c}{k}\right] \qquad 8.35$$

Use 8.35 to eliminate δ_c' from 8.33. Note that in matter dominated era $H \propto a^{-3/2}$

$$dn = \frac{dt}{a} = \frac{da}{a^2H}$$

$$\frac{d(aH)}{dn} = \frac{d(ka^{-1/2})}{dn} = -\frac{k}{2}\frac{da}{dn}a^{-3/2} = -\frac{k}{2}\sqrt{a}H = \frac{-k^2}{2a} = -\frac{a^2H^2}{2}$$

$$\frac{2}{3}\frac{k^2\Phi}{a^2H^2} - \frac{3aH\, iu_c}{k} = \delta_c$$

$$\frac{2}{3}\frac{k^2\Phi'}{a^2H^2} + \frac{2k^2\Phi}{3aH} - \frac{3aH\, iu_c'}{k} + \frac{3a^2H^2 iu_c}{2k} = \delta_c' \quad - (\cancel{8.36})$$

Subs in 8.33

$$\frac{2k^2\Phi'}{3a^2H^2} + \frac{2k^2\Phi}{3aH} - \frac{3aH\, iu_c'}{k} + \frac{3a^2H^2 iu_c}{2k} + iku_c = \Phi' \quad - (8.36)$$

→ Now there are two first order eq^{n}s in Φ & u_c (8.34, 8.36). We try to use them to form a single second order diff^n eq^n.

→ First eliminate u_c' from 8.36 using 8.34

$$\frac{2k^2\Phi'}{3a^2H^2} + \frac{2k^2\Phi}{3aH} - \frac{3aH\, i}{k}\left[ik\Phi - aHu_c\right] + \frac{3a^2H^2 iu_c}{2k} + iku_c = \Phi'$$

$$\frac{2k^2\Phi'}{3a^2H^2} + \left[\frac{iu_c}{k} + \frac{2\Phi}{3aH}\right]\left[k^2 + \frac{9a^2H^2}{2}\right] = \Phi' \quad - (8.37)$$

→ If the second order eq^n is of the form $a\Phi'' + b\Phi' = 0$ then Φ = const. is a solution to the eq^{n}s. So we differentiate 8.37 w.r.t. n & check if there are terms proportional to Φ.

→ Note that in this diff^n we're not concerned with the terms proportional to Φ' or Φ''.

Diff^n ~~w.r.t~~ 8.37 w.r.t η we get

$$\left(\frac{i u_c'}{k} + \frac{2\Phi}{3aH}\right)\left(\frac{9a^2H^2}{2} + k^2\right) + \left(\frac{i u_c}{k} + \frac{2\Phi}{3aH}\right)\left(\frac{9}{2}(-a^3H^3)\right)$$

↳ (From 8.34)

$$= \left(\frac{i}{k}\left(ik\Phi - aHu_c\right) + \frac{2\Phi}{3}\right)\left(\frac{9a^2H^2}{2} + k^2\right) + \frac{i\,aHu_c}{k}\left(-\frac{9}{2}a^2H^2\right)$$

$$- 3\Phi(a^2H^2)$$

$$= -\frac{i aHu_c}{k}\left[\left(\frac{9a^2H^2}{k}\right) \cancel{-\frac{aH}{k}} + k^2\right]$$

$$+ \cancel{\frac{2\Phi}{3}} \; -\frac{2\Phi}{3}\left(9a^2H^2 + k^2\right)$$

$$= -\left[\frac{i aHu_c}{k} + \frac{2\Phi}{3}\right]\left(9a^2H^2 + k^2\right) \quad - (8.38)$$

From 8.37 terms in square brackets is proportional to Φ'. Hence the eq^n doesn't have any term proportial to Φ. So Φ = const. is a sol^n.

→ Potentials remain constant as longe as the universe is matter dominated. When dark energy comes to dominate ($a \geq 0.1$) the potential starts to decay.

→ The main result of this section is "Transfer function defined in eq (8.2) is close to unity on all scales that enter the horizon after the universe becomes matter dominated". (For all $k << a_{eq}H(a_{eq})$)

→ This scale k_{eq} is calulated in Ex 8.5 soln.

8.3 Small Scales

→ In this section we solve for the modes which enter the horizon well into radiation-dominated era. The problem divides into (i) modes in the radiation era crossing the horizon (ii) Sub horizon modes passing through the epoch of equality.

→ We aren't able to handle (analytically) the modes that enter the horizon around the epoch of equality.

8.3.1 Horizon Crossing

→ During Radiation dominated era, radiation perturbation & the gravitation potential affect each other. Matter perturbations do not affect the potential but are driven by it.

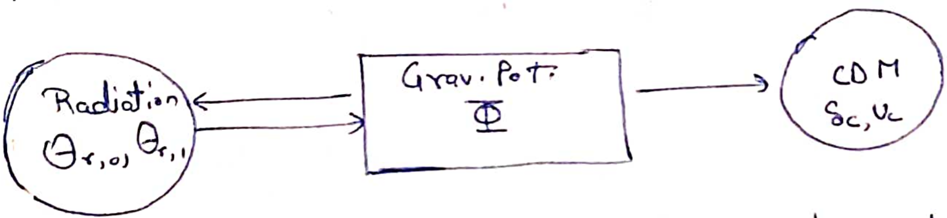

Fig. (8.5)

→ So we solve for matter perturbations in two steps, First we must solve for the coupled eq^ns for $\Theta_{r,0}$, $\Theta_{r,1}$ & Φ. Then we solve for matter evolution using the potential as an external driving force.

→ We choose eq (8.15) by while dropping v_c, δ_c terms.

$$\Phi = \frac{6a^2H^2}{k^2}\left[\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right] \quad -(8.40) \qquad \left\{\frac{8\pi\rho_r}{3a} = H^2\right\}$$

→ In radiation era $\eta = \int d\eta = \int \frac{da}{a^2H} = \int \cancel{\frac{da}{}} = \frac{a}{k} = \frac{1}{aH} = \eta$ $\qquad \left\{H = \frac{k}{a^2}\right\}$

→ Using this rel^n we can eliminate $\Theta_{r,0}$ & $\Theta'_{r,0}$ from the two eq^ns 8.10, 8.11

From e.40 we have

$$\Phi' = \underbrace{-\frac{12a^3H^3}{k^2}\left[\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right]}_{\left(-\frac{2\Phi}{\eta}\right)} + \frac{6a^2H^2}{k^2}\left[\Theta'_{r,0} + \frac{3aH}{k}\Theta'_{r,1} - \frac{3a^2H^2}{k}\Theta_{r,1}\right]$$

$$\boxed{\Theta'_{r,0} = \left(\Phi' + \frac{2\Phi}{\eta}\right)\frac{\eta^2k^2}{6} - \frac{3}{k\eta}\Theta'_{r,1} + \frac{3}{k\eta^2}\Theta_{r,1}} \qquad \boxed{\Theta_{r,0} = \frac{k^2\eta^2\Phi}{6} - \frac{3\Theta_{r,1}}{k\eta}}$$

$$\frac{-3\Theta'_{r,1}}{kn} + k\,\Theta_{r,1}\left[1+\frac{3}{k^2n^2}\right] = -\Phi'\left[1+\frac{k^2n^2}{6}\right] - \frac{\Phi k^2 n}{3} \quad -(8.41)$$

$$\Theta'_{r,1} + \frac{1}{n}\Theta_{r,1} = -\frac{k}{3}\Phi\left[1-\frac{k^2n^2}{6}\right] \quad -(8.42)$$

→ We've obtained two first order equations ~~in~~ for Φ & $\Theta_{r,1}$. We can use 8.42 to obtain one second order eqⁿ.

First eliminate $\Theta'_{r,1}$ in (8.41) using (8.42).

$$-\frac{3}{kn}\left[-\frac{k}{3}\Phi\left(1-\frac{k^2n^2}{6}\right) - \frac{1}{n}\Theta_{r,1}\right] + k\,\Theta_{r,1}\left[1+\frac{3}{k^2n^2}\right] = -\Phi'\left[1+\frac{k^2n^2}{6}\right] - \frac{\Phi k^2 n}{3}$$

$$\left(1+\frac{k^2n^2}{6}\right)\left(\Phi'+\frac{\Phi}{n}\right) + k\,\Theta_{r,1}\left[1+\frac{6}{k^2n^2}\right] = 0$$

$$\Phi' + \frac{\Phi}{n} = \frac{-6}{kn^2}\Theta_{r,1} \quad -(8.43)$$

→ Now we diffⁿ 8.43 ~~to get term~~ & eliminate terms of order $\Theta_{r,1}$ & $\Theta'_{r,1}$.

$$\Phi'' - \frac{\Phi}{n^2} + \frac{\Phi'}{n} = +\frac{12\,\Theta_{r,1}}{kn^3} - \frac{6}{kn^2}\Theta'_{r,1}$$

$$= \cancel{\frac{12}{kn^2}}\ -\frac{2}{n}\left(\Phi'+\frac{\Phi}{n}\right) - \frac{6}{kn^2}\left[-\frac{k}{3}\Phi\left(1-\frac{k^2n^2}{6}\right) + \frac{1}{n}\left(-\frac{kn^2}{6}\right)\left(\Phi'+\frac{\Phi}{n}\right)\right]$$

$$\Phi'' - \frac{\Phi}{n^2} + \frac{\Phi'}{n} = -\frac{3}{n}\left(\Phi'+\frac{\Phi}{n}\right) + \frac{2\Phi}{n^2} - \frac{k^2\Phi}{3}$$

$$\boxed{\Phi'' + \frac{4\Phi'}{n} + \frac{k^2\Phi}{3} = 0} \quad -(8.44)$$

– (Wave Eqⁿ in fourier space with a damping term due to expansion)

↳ Initial conditions are Φ = const.

→ We expect the solⁿ to be oscillating (damped).

→ Consider $u \equiv \Phi \eta$

$$\Phi = \frac{u}{\eta}$$

$$\Phi' = \frac{u'}{\eta} - \frac{u}{\eta^2} \qquad \Phi'' = \frac{u''}{\eta} - \frac{2u'}{\eta^2} + \frac{2u}{\eta^3}$$

$$\frac{u''}{\eta} - \frac{2u'}{\eta^2} + \frac{2u}{\eta^3} + \frac{4}{\eta}\left(\frac{u'}{\eta} - \frac{u}{\eta^2}\right) + \frac{k^2}{3}\frac{u}{\eta} = 0$$

$$u'' + \frac{2u'}{\eta} + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right)u = 0$$

We try to convert it into a form similar to spherical bessel's equations.

$$u'' + \frac{2u'}{\eta} + \frac{k^2}{3}\left(1 - \frac{2}{\frac{\eta^2 k^2}{3}}\right)u = 0$$

$$x = \frac{\eta k}{\sqrt{3}} \qquad \cancel{dx = \frac{d\eta k}{\sqrt{3}}}$$

$$u' = \frac{du}{d\eta} = \frac{du}{dx}\cdot\frac{dx}{d\eta} = u'\cdot\frac{k}{\sqrt{3}}$$

$$\frac{u'}{\eta} = \frac{u' k/\sqrt{3} \times k/\sqrt{3}}{x} = \frac{u' k^2}{3x}$$

$$u'' = \frac{du'}{dx}\cdot\frac{dx}{d\eta}\cdot\frac{k}{\sqrt{3}} = u''\cdot\frac{k^2}{3}$$

$$\Rightarrow u'' + \frac{2u'}{x} + \left(1 - \frac{2}{x^2}\right)u = 0$$

(for $x = k\eta/\sqrt{3}$)

↳ Spherical bessel eq^n of order 2. Two solutions $j_1(k\eta/\sqrt{3})$ & $n_1(k\eta/\sqrt{3})$. $n_1(k\eta/\sqrt{3})$ blows up at $\eta = 0$ so it is discarded. (Spherical Bessel sol^n) (Spherical Neumann sol^n)

since our initial cond^n is Φ = const.

$$\boxed{\Phi(\vec{k}, \eta) = 2\left(\frac{\sin x - x\cos x}{x^3}\right)_{x = k\eta/\sqrt{3}} R(\vec{k})} \qquad (8.46)$$

bcoz $\Phi(\vec{k}, \eta)|_{\eta \to 0} = \frac{2}{3} R(\vec{k})$

→ A nice interpretation of these oscillations is given in the two paras above & below eq (8.47).

→ Once we get the sol^n for potential Φ we can determine the evolution of matter perturbations. (Right part of Fig 8.9)

→ We'll use eq^{n}s 8.12 & 8.13.

Differentiating 8.12 ~~we get~~ & using 8.13 we get

$$\delta_c'' + ik\left(-\frac{a'}{a}v_c + ik\Phi\right) = -3\Phi''$$

$$\delta_c'' + ik\cancel{\left(-\frac{a'}{a}\right)} - \frac{a'}{a}\left(-3\Phi' - \delta_c'\right) - k^2\Phi = -3\Phi''$$

$$\delta_c'' + aH\,\delta_c' = -3\Phi' + k^2\Phi - 3\Phi' aH$$

{ $aH = \frac{1}{\eta}$ in rad. domination }

$$\boxed{\delta_c'' + \frac{1}{\eta}\delta_c' = S(k,\eta)} \quad - (8.49)$$

Two homogenous $\left(\delta_c'' + \frac{\delta_c'}{\eta} = 0\right)$ solutions are

$$\boxed{S(k,\eta) = -3\Phi'' + k^2\Phi - 3\frac{\Phi'}{\eta}} \quad (8.50)$$

↳ source term

$S_1(\eta) = \delta_c = \text{const}$ · $S_2(\eta) = \delta_c = \ln\eta$

→ Since we don't know the particular sol^n, we can construct it using the Greene's function $G(\eta,\tilde{\eta})$

$$G(\eta,\tilde{\eta}) = \frac{S_1(\eta)S_2(\tilde{\eta}) - S_1(\tilde{\eta})S_2(\eta)}{\underbrace{S_1'(\tilde{\eta})S_2(\tilde{\eta})}_{0} - S_1(\tilde{\eta})S_2'(\tilde{\eta})} = \frac{\ln k\tilde{\eta} - \ln k\eta}{-1/\tilde{\eta}}$$

{ k is added for convenience }

$$= -\tilde{\eta}\left[\ln k\tilde{\eta} - \ln k\eta\right]$$

$$\rightarrow \delta_c(k,\eta) = C_1 + C_2\ln(k\eta) - \int_0^{\eta} d\tilde{\eta}\, S(k,\tilde{\eta})\left[(\tilde{\eta})(\ln(k\tilde{\eta}) - \ln(k\eta))\right] \quad -(8.51)$$

↳ Initial conditions $\delta_c(k,\eta)|_{\eta\to 0} = \text{const}$.

$$\Rightarrow C_2 = 0$$

→ Now we consider the integrale $\int_0^{\eta} d\tilde{\eta}\, S(k,\tilde{\eta})\left[(\tilde{\eta})(\ln k\tilde{\eta} - \ln k\eta)\right]$

If $\eta >$ horizon crossing time (i.e. $k\eta \gg 1$) then the potential & the source terms $S(k,\tilde{\eta})$ decays after horizon crossing. Thus $S(k,\tilde{\eta})\ln(k\tilde{\eta})$ will just asymptote to some constant. Similarly $S(k,\tilde{\eta})\,\tilde{\eta}\,\ln k\eta$ will lead to a term proportional to $\ln(k\eta)$. Thus ~~for we~~ after the mode has entered the horizon we expect

$$\delta_c(k,\eta) = AR\ln B + AR\ln(k\eta) = AR\ln(Bk\eta)$$

$$\rightarrow \quad AR \ln(B) = C_1 - \int_0^\infty d\tilde{n}\, S(k,\tilde{n})\, \tilde{n}\, \ln(k\tilde{n})$$ (17)

where $C_1 = \delta_c|_{n\to 0} = R$

$$\rightarrow \quad AR = \int_0^\infty d\tilde{n}\, S(k,\tilde{n})\tilde{n}$$

→ Using 8.46 & 8.50 these integrals can be evaluated & A & B are obtained as 6.0, 0.62 respectively.

→ Evolution of $\delta_c(k,n)$ can be seen in fig. 8.11. Note that the perturbations in dark matter density grow in radiation era in (contrast to those in the radiation (& baryon) components which decay and oscillate. (Para just after eq 8.47)

→ The reason for this growth is that CDM does not have any pressure to counteract the effect of gravity. (Recall $P_{matter} = 0$). The growth is not as prominent as during the matter era (where the growth is due to const. potential Φ is $\delta_c \propto a$ while in radiation ~~era~~ era $\delta_c \propto \ln(a)$) due to the more rapid expansion of universe during radiation dom. but it still exists.

8.3.2 Sub Horizon Evolution

→ See Fig (8.12) (& observe ~~simi~~ the diff. from fig (8.8)) to consider the regime. Although initially the potential Φ is determined by the radiation, eventually the growth in matter perturbations (δ_c term in 8.15) more than offsets the higher mean radiation density ($\rho_r \Theta_{r,0}$ term in 8.15). once this happens (even if in radiation dominated phase) the grav. potential & dark matter perturbations evolve together & do not care what happens to the radiation.

→ We'll use eq^n (8.12, 8.13, 8.15) to ~~no~~ construct a 2nd order diff^n equation. Since we'll follow the evolution through the transition epoch (a_{eq}) it is convenient to use $y = a/a_{eq} = (\delta_c/\rho_r)$

$\rightarrow \quad \frac{d}{d\eta} = aHy\frac{d}{dy}$. ~~We are going to us~~

$$aHy\frac{d\delta_c}{dy} + iku_c = -3aHy\frac{d\Phi}{dy} \Rightarrow \boxed{\frac{d\delta_c}{dy} + \frac{ik}{aHy}u_c = -3\frac{d\Phi}{dy}} \quad - (8.55)$$

$\rightarrow \quad \frac{a'}{a} = aH$

$$\boxed{\frac{du_c}{dy} + \frac{u_c}{y} = \frac{ik}{aHy}\Phi} \quad - (8.56)$$

$\rightarrow$ ~~$k^2\Phi$~~ In eq^n 8.15 $\rho_r\Theta_{r,0}$, $\rho_r\Theta_{r,1}$ are obv. neglected.
Also since we're well with in the horizon, $k\eta \gg 1$, ($k \gg aH$)
Hence we're left with

$$k^2\Phi = 4\pi G a^2 \rho_c \delta_c \quad - (8.15)$$

Using ~~$\frac{8\pi\rho_c}{3} = H^2$~~ (neglecting both $\underbrace{\rho_b}_{\text{(Results slightly deflected from numerical calc.)}}$ & $\underbrace{\text{dark energy}}_{\text{(We're working in such regime)}}$)

~~$k^2\Phi \Rightarrow$~~ $\frac{8\pi}{3}(\rho_c+\rho_r) = H^2$

$\frac{8\pi}{3}\rho_c\left(1+\frac{1}{y}\right) = H^2$

$$\boxed{k^2\Phi = \frac{3y}{2(y+1)}a^2H^2\delta_c} \quad - (8.56)$$

$\rightarrow$ Diff^n 8.55 w.r.t y

$$\frac{d^2\delta_c}{dy^2} + \frac{ik}{aHy}\left[\frac{ik\Phi}{aHy} - \frac{u_c}{y}\right] + iku_c\frac{d}{dy}\left(\frac{1}{aHy}\right) = -\frac{3d^2\Phi}{dy^2}$$

~~$aHy = a\frac{a'}{a}\frac{a}{a_{eq}} = \frac{a^2}{a_{eq}}\ldots (a_{eq})y^2\sqrt{\frac{8\pi\rho_c}{3}\left(\frac{y+1}{y}\right)} = aHy$~~

~~$\frac{d}{dy}(aHy) = aH + y\frac{d}{dy}(aH) = aH + y\frac{d}{dy}\left(\frac{a'}{a}\right)$~~

~~$\rightarrow \frac{d(aHy)}{dy} = 2y\left[\;\right] + y^2\sqrt{\frac{8\pi\rho_c}{3}}\frac{1}{2\sqrt{\frac{y+1}{y}}}\left(-\frac{1}{y^2}\right)$~~

~~$= \frac{2}{y}(aHy) - \frac{1}{2\left(\frac{y+1}{y}\right)}\frac{aHy}{y^2} = 2aH - \frac{aH}{2(y+1)}$~~

~~$\rightarrow \frac{d}{dy}\left(\frac{1}{aHy}\right) = -\frac{1}{a^2H^2y^2}\frac{aH(4y+3)}{2(y+1)} = \frac{-(4y+3)}{2aHy^2(y+1)} = \frac{aH(4y+3)}{2(y+1)}$~~

→ $\cancel{\frac{d^2\delta_c}{dy^2}}$ $\frac{d}{dy}\left(\frac{1}{aHy}\right) = \frac{-1}{y^2a^2H} - \frac{1}{a^2H^2}\frac{d}{dy}\left(\dot{a}\right)$

$$= -\frac{1}{y^2aH} - \frac{1}{a^2H^2}\frac{1}{aHy}\frac{d}{d\eta}\left(\dot{a}\right) = -\frac{1}{y^2aH} - \frac{1}{a^2H^2}\cdot\frac{1}{Hy}\ddot{a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_c+\rho_r+3P_c+3P_r\right) = -\frac{4\pi G}{3}\left(\rho_c+\rho_r+\rho_r\right) \quad \left\{ P_r = \frac{\rho_r}{3},\ P_c = 0 \right\}$$

$$= -\frac{4\pi G}{3}\rho_c\left(1+\frac{2}{y}\right) = -\frac{4\pi G}{3}\rho_c\left(\frac{y+2}{y}\right)$$

$$= -\frac{1}{2}\left(\frac{y+2}{y+1}\right)H^2$$

$$\Rightarrow \frac{d}{dy}\left(\frac{1}{aHy}\right) = \frac{-1}{y^2aH} + \frac{1}{a^2\cancel{H^2}}\cdot\frac{1}{Hy}\frac{\cancel{H^2}}{2}\left(\frac{y+2}{y+1}\right)a$$

$$= \frac{1}{yaH}\left[\frac{y+2}{2y+2} - \frac{1}{y}\right] = \frac{1}{aHy}\left[\frac{y^2-2}{2y(y+1)}\right]$$

→ $$\frac{d^2\delta_c}{dy^2} - \frac{ik(2+3y)\,u_c}{2aHy^2(1+y)} = -\frac{3d^2\Phi}{dy^2} + \frac{k^2\Phi}{a^2H^2y^2} \quad - (8.58)$$

We can write u_c from 8.55 by neglecting Φ at sub-horizon scales.

This term can be neg.

($k/aH >> 1$)

$$\frac{d^2\delta_c}{dy^2} + \frac{2+3y}{2y(y+1)}\frac{d\delta_c}{dy} - \frac{3\delta_c}{2y(y+1)} = 0 \quad \left\{ k^2\Phi \text{ in RHS} = \frac{3y}{2(y+1)}a^2H^2\delta_c \right\}$$

Meszaros eq^n

→ We need to obtain two independent solutions & use initial conditions $\delta_c \propto \ln(k\eta)$.

→ For first sol^n we'll use the result (which we'll prove in 8.5) that δ_c grows with a in deep matter era.

$\Rightarrow \frac{d^2\delta_c}{da^2} = 0$ for that mode.

$\rightarrow \frac{\delta'_{c+}}{\delta_{c+}} = \frac{3}{2+3y}$ $\quad \underline{\delta_{c,+} \propto y + 2/3} \Rightarrow$ $\boxed{D_+(a) = a + \frac{2a_{eq}}{3}}$ ($a \gg a_{eq}$, $D_1(a) = a$)

Describes this growing mode

Scale independent growth factor in deep matter epoch. (Refer section 8.1)
(Since we've neglected dark energy it is valid only till $a < 0.1$)

→ For finding the second solution we use the variable

$$U \equiv \frac{\delta_c}{y+2/3} \qquad \delta_c = (y+2/3)\,U$$

$$d\delta_c/dy = (y+2/3)\,dU/dy + U$$

$$\frac{dU}{dy} = \frac{d\delta_c/dy}{y+2/3} - \frac{\delta_c}{(y+2/3)^2} \qquad \frac{d^2 U}{dy^2} \qquad \frac{d^2\delta_c}{dy^2} = \left(y+\frac{2}{3}\right)\frac{d^2U}{dy^2} + 2\frac{dU}{dy}$$

$$\left(y+\frac{2}{3}\right)\frac{d^2U}{dy^2} + 2\frac{dU}{dy} + \frac{3(y+2/3)}{2y(y+1)}\left[\left(y+\frac{2}{3}\right)\frac{dU}{dy} + U\right] - \frac{3(y+2/3)U}{2y(y+1)} = 0$$

$$\frac{3}{2}\left(1+\frac{3y}{2}\right)\frac{d^2U}{dy^2} + \frac{dU}{dy}\left[\frac{4y^2+4y+3(y^2+4/9+4y/3)}{2y(y+1)}\right] = 0$$

$$\left(1+\frac{3y}{2}\right)\frac{d^2U}{dy^2} + \frac{(21/4)y^2+6y+1}{y(y+1)}\frac{dU}{dy} = 0 \qquad \text{— (8.61)}$$

$$\frac{U''}{U'} = -\frac{21y^2+24y+4}{2y(y+1)(3y+2)} = -\frac{1}{y} - \frac{1}{2(y+1)} - \frac{6}{(3y+2)}$$

$$\underline{U' \propto y^{-1}(y+1)^{-1/2}(y+2/3)^{-2}}$$

→ Integrating again we obtain

$$D_-(y) = (y+2/3)\ln\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1+y}$$

At early times $y \ll 1$ $D_+ = \text{const.}$, $\underline{D_- \propto \ln y}$. At late times ($y \gg 1$), the growing sol^n D_+ scales as y while decaying mode D_- falls off as $y^{-3/2}$.

→ The general solution to the Meszaros eq^n is therefore (8.64)

$\delta_c(k,y) = C_1 D_+(y) + C_2 D_-(y)$ $(y \gg y_H)$

$y_H = a_H/a_{eq}$ → epoch at which k = comoving hubble radius (mode enters horizon)

→ From eq 8.6 we see that for $k \gg k_{eq}$ (small scales) $a_H \ll a_{eq}$. (i.e. y_H is itself much less than 1)

→ ~~To det. We'll use~~ We can use eq^n(8.52) in the regime $\underbrace{y_H \ll y \ll 1}$ (Within the horizon but before equality) to obtain a rel^n b/w C_1 & 2.

→ We'll do this for modes which enter the horizon before equality. (by matching the two sol^{n}s in 8.64 & 8.52 & their first derivs.)

$$8.65(a) - AR \ln(By_m/y_H) = C_1 D_+(y_m) + C_2 D_-(y_m)$$

{ where y_m is matching epoch $y_H \ll y_m \ll 1$ }

$$8.65(b) - \frac{AR}{y_m} = C_1 D_+'(y_m) + C_2 D_-'(y_m)$$

{ Also replacing $k\eta$ with y/y_H is valid only in radiation era }

These two eq^{n}s determine C_1 & C_2.

→ Fig 8.13 shows the comparison of analytic solutions with the numerical ones.

8.4 The Transfer Function

of dark matter perturbations

→ We'll use the analytic solutions, derived in sections (8.2 & 8.3) to ~~obtai~~ get the transfer functions.

→ Eq^n (8.7) gives us an expression for matter (or CDM) overdensity in terms of $T(k)$ & $D_+(a)$ at late times & for ~~small~~ small scale modes.

(~~that~~ deep in matter dom. epoch)

→ We can get the same using the growing mode sol^n $D_+(a)$ in eq^n (8.64). (Since at later times the $D_-(a)$ sol^n would have already died down).

→ So we determine C_1

$$AR \ln(By_m/y_H)\, D_-'(y_m) - \frac{AR}{y_m} D_-(y_m) = C_1 \left[D_+(y_m) D_-'(y_m) - D_+'(y_m) D_-(y_m) \right]$$

$D_+(y_m) = a_{eq}(y_m + 2/3)$ $\quad D_-(y_m) = \left(y_m + \frac{2}{3}\right)\ln\left[\frac{\sqrt{1+y_m}+1}{\sqrt{1+y_m}-1}\right] - 2\sqrt{1+y_m}$ (22)

$D_+'(y_m) = a_{eq}$ $\quad D_-'(y_m) = \ln\left(\frac{\sqrt{1+y_m}+1}{\sqrt{1+y_m}-1}\right) + \left(y_m+\frac{2}{3}\right)\frac{\sqrt{1+y_m}-1}{\sqrt{1+y_m}+1} \times \frac{\frac{(\sqrt{1+y_m}-1)}{2\sqrt{1+y_m}} - \frac{(\sqrt{1+y_m}+1)}{2\sqrt{1+y_m}}}{(\sqrt{1+y_m}-1)^2}$

$D_-'(y_m) = \ln\left(\frac{\sqrt{1+y_m}+1}{\sqrt{1+y_m}-1}\right) - \frac{(y_m+2/3)}{y_m\sqrt{1+y_m}}$

$y_m << 1$

$D_-'(y_m) = \frac{-2}{3y_m}$ $\quad D_+'(y_m) = a_{eq}$

$D_-(y_m) = \frac{2}{3}\ln\left(\frac{4}{y_m}\right) - 2$ $\quad D_+(y_m) = \frac{2}{3}a_{eq}$

$D_+D_-' - D_+'D_- = \frac{-4}{9}\frac{a_{eq}}{y_m} - \frac{2a_{eq}}{3}\ln\left(\frac{4}{y_m}\right) + 2a_{eq}$

$y_m << 1$

$\simeq -\frac{4}{9}\frac{a_{eq}}{y_m}$ or $\cancel{\frac{4}{3y_m}}$

$$C_1 \to -\frac{9}{4}\frac{AR}{a_{eq}}\left[-\frac{2}{3}\ln\left(\frac{By_m}{y_H}\right) - \frac{2}{3}\ln\left(\frac{4}{y_m}\right) + 2\right]$$

$$\delta_c(\vec{k}, a) = C_1 D_+(a) \quad (a >> a_{eq})$$

$$\boxed{\delta_c(\vec{k}, a) = \frac{3}{2}\frac{AR(\vec{k})}{a_{eq}}\ln\left[4Be^{-3}\frac{a_{eq}}{a_H}\right]D_+(a)} \quad - (8.68)$$

On very small scales $k >> k_{eq}$ we can even sub. $\frac{a_{eq}}{a_H} = \frac{\sqrt{2}k}{k_{eq}}$.

→ Comparing with eq (8.7) we have

Note that eq 8.7 was for δ_m so deviations from ($k >> k_{eq}$) this $T(k)$ are naturally expected.

$$\boxed{T(k) = \frac{15}{4}\frac{\Omega_m H_0^2}{k^2 a_{eq}} A\ln\left[4Be^{-3}\frac{\sqrt{2}k}{k_{eq}}\right]}$$

→ From Ex 8.5 we have $k_{eq} = \sqrt{2\Omega_m}H_0 a_{eq}^{-1/2}$. Expressing a_{eq} in the d^r in terms of k_{eq} & plugging in values of A & B we obtain. ($A = \cancel{0.6}$, B 6.4, $B = 0.44$)

$$\boxed{T(k) = 12.0\frac{k_{eq}^2}{k^2}\ln\left[0.12\frac{k}{k_{eq}}\right]} \quad (k >> k_{eq}) \quad (8.71)$$

↳ Note that this is accurate on very small scales $k \geq 1h$ Mpc^{-1}

→ Using eqⁿ ~~8.8~~ 8.8 we can track the matter power spectrum as a function of k. (Fig. 8.14) As already noted on pg (4), $T(k)$ for $k >> k_{eq}$ decreases as k increases & hence power spectrum decreases. Since $k_{eq} \propto \sqrt{\Omega_{m0}}$; for higher Ω_{m0} the power spectrum turns at higher k values.

→ Nice discussion after Fig 8.14 about $\Omega_m h^2$.

→ Note that the following physical effects have been neglected in analytic treatment

(i) No anisotropic stress ($\Phi = -\Psi$) (Θ_2, N_2 & higher order terms are negligible). Dropping this assumption changes the factor of 9/10 by 0.86. This leads to a rise in the small scale transfer function.

(ii) We've also ~~assume~~ neglected the effect of Baryons. This would be taken into account in section 8.6

(iii) Growth factor at late times (when Dark Energy become important)

8.5 The Growth Factor

→ At late times, if ~~the~~ dark energy were not present & neutrinos didn't get mass, Mesazaros eq^n would've worked.

→ The timeline that is under consideration here is ~~just~~ after decoupling ($z \simeq 1100$) where ($z_{eq} \simeq 3400$). At this time $P_{Baryons} = P_{dar\,matter} = 0$. So for baryons also we can use eq^n. 8.12 – 8.13.

→ $\rho_m \delta_m = \rho_c \delta_c + \rho_b \delta_b$, $\rho_m v_m = \rho_c v_c + \rho_b v_b$

→ $a \times$ eq (8.12)

$$a\delta_c' + ika v_c = -3a\Phi'$$

$$a\delta_c'' + a'\delta_c' + ika' v_c + ika v_c' = -3a'\Phi' - 3a\Phi''$$

(from 8.13)

$$a\delta_c'' + a'\delta_c' + ak^2\Phi = \underbrace{-3a'\Phi' - 3a\Phi''}_{\text{Can be neglected as } \Phi \sim \text{constant on sub horizon scale}}$$

$$\boxed{(a\delta_m')' = ak^2\Phi(\vec{k},\eta)} \quad -(8.72)$$

(c replaced by m)

→ We'll replace $k^2\Phi$ on RHS using eq^n 8.14. Note that $\Theta_{r,0}$, $N_{r,0}$ can be neglected due to matter dominated regime & $k^2 \gg (a'/a)^2\Phi$ $(aH)^2\Phi$ can be neglected in comparison to $k^2\Phi$ as horizon size is quite large.

(wouldn't this assumption become invalid?) (D)

$$k^2\Phi(\vec{k},\eta) = 4\pi G a^2 \rho_m(\eta)\delta_m(\vec{k},\eta)$$

Hence we get

using $\frac{8\pi G\rho_m}{3} = H^2 = \frac{H_0^2 \Omega_{m_0}}{a^3}$ (matter dominated regime)

$$4\pi G\rho_m = \frac{3H_0^2\Omega_m}{2a^3}$$

→ $(a\delta_m')' = \frac{3}{2}\Omega_{m_0} H_0^2 \delta_m$

→ Changing time variable from η to a.

$$\frac{d\delta_m}{d\eta} = \frac{d\delta_m}{da}\cdot\frac{da}{d\eta} = \frac{d\delta_m}{da}\cdot\frac{dt}{d\eta}\frac{da}{dt} = \frac{d\delta_m}{da} a^2 H$$

$$a\frac{d\delta_m}{d\eta} = a^3 H\frac{d\delta_m}{da},\quad \frac{d}{d\eta}\left(a\frac{d\delta_m}{d\eta}\right) = a^2H\frac{d(a^3H)}{da}\frac{d\delta_m}{da} + (a^2H)(a^3H)\frac{d^2\delta_m}{da^2}$$

$$\rightarrow \frac{d^2\delta_m}{da^2} + \frac{d\ln(a^3H)}{da}\frac{d\delta_m}{da} - \frac{3\Omega_m H_0^2\delta_m}{2a^5H^2} = 0$$ (24)

↳ This eqn can be solved numerically
However as instructed in exercise 8.8 we solve it for a standard ΛCDM model.

Ex 8.8
(a) $\delta_m = CH$ $\quad C = \text{const.}$

$$c\frac{d^2(H)}{da^2} + \underbrace{\frac{d(3\ln a + \ln H)}{da}\cdot c\frac{dH}{da}} - \frac{3\Omega_m H_0^2 \cancel{\delta_m}(cH)}{2a^5 \cancel{H}^2}$$

$$c\frac{d^2H}{da^2} + \frac{3c}{a}\frac{dH}{da} + \frac{c}{H}\left(\frac{dH}{da}\right)^2 - \frac{3\Omega_m H_0^2 c}{2a^5 H}$$

$$c\frac{d}{da}\left(a^3\frac{dH}{da}\right) + \frac{ca^3}{H}\left(\frac{dH}{da}\right)^2 - \frac{3\Omega_m H_0^2 c}{2a^2 H}$$

$$cH\frac{d}{da}\left(a^3\frac{dH}{da}\right) + ca^3\left(\frac{dH}{da}\right)^2 - \frac{3\Omega_m H_0^2 c}{2a^2}$$

$$2H\frac{dH}{da} = H_0^2\left[-3\frac{\Omega_{m0}}{a^4} \cancel{+}2\frac{(1-\Omega_m-\Omega_\Lambda)}{a^3}\right]$$

$$a^3\frac{dH}{da} = \frac{H_0^2}{2H}\left[-\frac{3}{2}\frac{\Omega_{m0}}{a} - 2(1-\Omega_m-\Omega_\Lambda)\right]$$

$$\frac{d}{da}\left(a^3\frac{dH}{da}\right) = -\frac{H_0^2}{2H^2}\left(\quad\right)\left(\frac{dH}{da}\right) + \frac{H_0^2}{2H}\left[\frac{3\Omega_{m0}}{2a^2}\right]$$

$$cH\frac{d}{da}\left(a^3\frac{dH}{da}\right) = -\frac{cH_0^2}{\cancel{2H}}\frac{\cancel{2H}}{\cancel{H_0^2}}a^3\left(\frac{dH}{da}\right)^2 + \frac{cH_0^2}{2}\left[\frac{3\Omega_{m0}}{2a^2}\right]$$

$\Rightarrow$ $\delta_m = CH$ is a solution. So this soln is not feasible.
as $a\uparrow$ $H\downarrow$ & hence $\delta_m\downarrow$.
→ Rest soln on the other page.

$(\Lambda,$ curvature$)$

$$\rightarrow D_+(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{da'}{(a'H(a')/H_0)^3}$$
↳ valid only when Ω_Λ is const.

→ For other dark energy models we've ~~the parameter~~ an empirical formula
$$\boxed{f(a) = \frac{d\ln D_+(a)}{d\ln a} \sim [\Omega_m(a)]^{0.55}}$$

$$\Omega_m(a) = \frac{8\pi G}{3}\frac{\delta_m(a)}{H^2(a)}$$

at $a=1$ $\Omega_m(a) = \Omega_m$ (or Ω_{m0})

→ Fig 8.15 summarises this nicely.

8.6 Beyond CDM & Radiation

→ First we'll consider the effect of Baryons which const. (16% of the total matter)
→ Then we'll consider effect of neutrino masses
→ Finally Dark Energy would be considered

8.6.1 Baryons

→ We observe Fig 8.16 we draw an important conclusion
→ The transfer function is suppressed relative to the no-baryon case on small scales. This can be explained in two stages. **Before decoupling**: Rad^n perturb. do not grow inside the horizon, the baryon densities also then don't grow as they are coupled. Hence $T(k)$ is actually less than the estimated one as 16% of matter now doesn't evolve. **After decoupling**:
Refer Dodelson

→ There is one more effect of Baryons: In fig 8.16 it can be seen that Baryons lead to small oscillations in the transfer function around $k \simeq 0.1 h\,Mpc^{-1}$. These oscillations are remanants of oscillations that the combined baryon-photon fluid enperiences before decoupling. ~~These~~ The oscillation in the potential (Eqn 8.46) are reflected in density of the baryon-photon fluid which are accoustic plasma waves. For this reason they are known as BAO.

→ ~~The~~ These oscillations ~~are~~ have very small amplitudes which is simply bcoz Baryons are such small fraction of total matter. The effect is much more pronounced in radiation (Next chapter). (26)

→ ~~We~~ Now we want to examine in a little more depth how does the baryons trace dark matter after decoupling.

We have $\delta_s' + ikU_s = -3\Phi'$

$$U_s' + \frac{a'}{a}U_s = ik\Phi \qquad (s = \{b, c\}) \qquad (8.79)$$

Define relative density perturb', relative velocity b/w baryons & CDM :

$$\delta_{bc} = \delta_b - \delta_c$$

$$U_{bc} = U_b - U_c$$

We can easily obtain

$$\delta_{bc}' + ik U_{bc} = 0$$

$$U_{bc}' + \frac{a'}{a}U_{bc} = 0$$

Solutions (1) $\delta_{bc} = C_s$, $U_{bc} = 0$

↓ Const. relative density perturb.

↓ How does it keep total matter fixed

(2) $U_{bc} = C_u/a$, $\delta_{bc} \propto C_u \int d\eta/a$

↓ An initial push to baryon

Both modes (1) & (2) are const. or decaying so they become insignificant in comparison to growing δ_m.

8.6.2 Massive Neutrinos

→ There are two effects that the mass of neutrinos produce. matter power spectrum may enable us Accurate measurement of to infer Neutrino masses.

(iii) ~~Power~~ Growth factor becomes dependent on k.

(i) Dodelson

(ii) As neutrinos stream fast, they stream out of high density region & damp the growth of small ~~str~~ scale structure. Perturbation on scales smaller than the typical distance neutrinos travel, the free-streaming scale are therefore suppressed.

$$k_{fs}(a) \simeq 0.063\,h\,Mpc^{-1}\,\frac{m_\nu}{0.1\,eV}\,\frac{a^2H(a)}{H_0} \qquad (8.83)$$

(i) ↳ m_ν ↑, neutrinos const. more of total density, more supression of small scale power.

(ii) At very large scales power spectrum of 0.06 eV is lesser than 0.2 eV