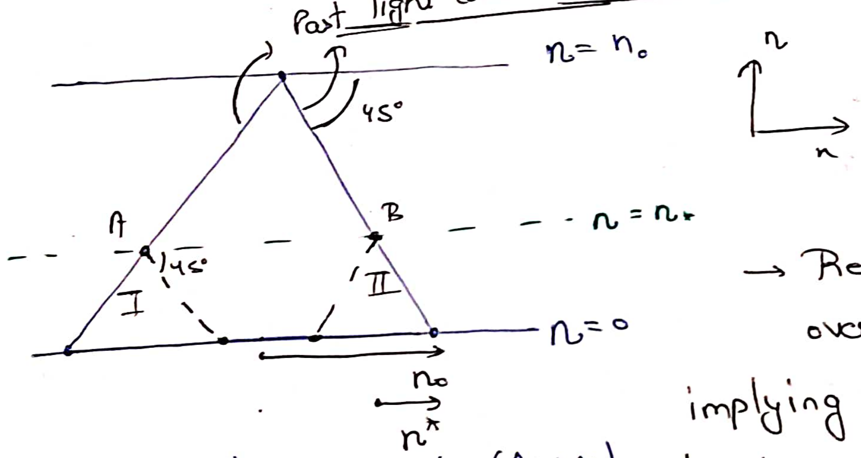
Ch-7 Initial Conditions

7.1 Horizon problem & a solution

1. <u>Comoving distance</u> η is the distance (~~como~~ on comoving grid) that light could travel (in absence of interactions) since $t=0$. $\boxed{\eta(t) = \int_0^t \frac{dt'}{a(t')}}$

→ No information can propogate further on coordinate grid than η since the beginning of time. Regions greater than η comoving distance are causally disconnected.



(On comoving grid ~~coords~~ light cones are at 45°)

→ Regions I & II don't overlap coz $\eta_0 >> 2\eta_*$ implying A & B's light cones don't overlap & hence they are causally disconnected.

→ Using concordance model (ΛCDM) we get $\eta_* = \eta(a_*) = 281\, h^{-1}$ Mpc & $\eta_0 \approx 14200 h^{-1}$ Mpc which gives us $\left(\frac{\eta_0}{\eta_*}\right)$ so causally disconnected regions.

→ Comoving distance b/w two patches separated by an angle θ is $\kappa(\theta) \simeq \kappa_*(\theta) = (\eta_0 - \eta_*)\theta$

For ~~H~~ $(\eta_0 - \eta_*)\theta \geq \eta_*$ the two regions were causally disconnected

$\Rightarrow \boxed{\theta \geq 1.2^\circ}$

→ Consider $n(a) = \int_0^a d\ln a' \frac{1}{a' H(a')}$

$$n(t) = \int_0^t \frac{dt}{a'(t)} = \int_0^{dt} \frac{dt}{da'} \frac{da'}{a'(t)} = \int_0^a \frac{d(\ln a')}{a' H(a')} = n(a)$$

$\frac{1}{aH}$ → Comoving hubble radius → equal to the distance light can travel in a time when $a \to ae$

→ gives a measure to judge whether particles can communicate within one e-fold of expansion at the given epoch.

→ n is nothing but logarithmic integral of comoving hubble radius

→ A/c to the matter or radiation dominated models H scales as $a^{-3/2}$ or a^{-2} resp. & hence hubble radius always increases & n receives major contribution from recent times.

→ But in case of inflation Hubble radius is quite large in beginning & hence the regions were in causal contact at the time of recombn.

→ Nice discussion after eq (7.4)

→ Generation of perturbations during inflation

→ $(k/2\pi)^{-1}$ is approximately the comoving wavelength of a perturbation length scale of that perturbation.

→ An important epoch is when comoving wavelength becomes of the order of Hubble radius $(1/aH)$. The mode k enters the horizon as it goes from $k \ll aH$ to $k \geq aH$, since it becomes an observable perturbation for an observer living in the universe.

7.2 Inflation

→ The simplest possibility to generate such a transitionary epoch of accelerated expansion is via the potential energy of a scalar field

→ For an accelerated expansion ($\ddot{a}>0$) negative pressures are required $\left[\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_s + \frac{3P_s}{c^2}\right)\right]$ ($\because \rho_s > 0$ always).

→ For matter $P \geq 0$, For radiation $P = \rho/3$.

→ Hence we try to check if a scalar field $\phi(\vec{x}, t)$ can have negative $\rho + 3P$.

Why does $\boxed{\frac{\partial(g^{\alpha\beta})}{\partial g^{\mu\nu}} = \delta^\alpha_\mu \delta^\beta_\nu}$??

$$T_{\mu\nu} = \frac{\delta L_\phi}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_\phi = \frac{\delta}{\delta g^{\mu\nu}}\left[-\frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - V(\phi)\right]$$

$$-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - g_{\mu\nu}V(\phi)$$

$$T_{\mu\nu} = -\frac{\partial\phi}{\partial x^\mu}\cdot\frac{\partial\phi}{\partial x^\nu} - \frac{1}{2}g_{\mu\nu}\left[g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} + V(\phi)\right]$$

$$\boxed{T^\alpha{}_\beta = +g^{\alpha\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^\beta} - \delta^\alpha_\beta\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi)\right]} \quad - (7.6)$$

Correct finally

↳ $V(\phi)$ is the potential for the field. For example a free field with mass 'm' has potential $V(\phi) = \frac{m^2\phi^2}{2}$.

→ We'll assume that the field is homogenous to the zeroth order, consisting of zeroth order part $\phi(t)$ & a first order perturbation $\delta\phi(\vec{x}, t)$. ??

Homogenous field $\phi(t)$ (only time derivs are relevant)

$$\boxed{T^\alpha{}_\beta = \underbrace{g^{\alpha 0}}_{-1\,\delta^\alpha_0}(\dot\phi)^2\delta^\beta_0 - \delta^\alpha_\beta\left[\frac{1}{2}\dot\phi^2 - V(\phi)\right]}$$

→ The time-time comp. $T^0_0 = -\rho$

$$T^0_0 \Rightarrow \boxed{\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)}$$ → if we think of $\phi(t)$ as $x(t)$, then dynamics of a single particle moving in a potential are recovered.

kinetic energy density of the field

→ $$\boxed{T^i_i = P = \frac{1}{2}\dot{\phi}^2 - V(\phi)}$$

A field config. with negative pressure is the one with more PE than KE

→ $$\boxed{w = \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\phi} - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}}$$ → eqn of state → should be close to -1.

→ Applying conservation of stress-energy tensor, $\nabla_\mu T^{\mu\nu} = 0$ & using (2.56) we get

$$\frac{\partial \rho}{\partial t} + 3H[\rho + P] = 0$$

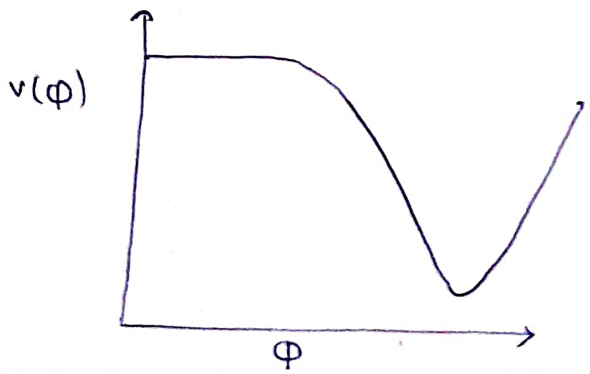
$$\dot{\phi}\ddot{\phi} + \frac{\partial V}{\partial \phi}\dot{\phi} + 3H\dot{\phi}^2 = 0$$

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0}$$ → (Klein Gordon Eqn)

Using conformal time η $dt = a\,d\eta$

$$\dot{\phi} = \frac{\phi'}{a}, \quad \ddot{\phi} = \cancel{\left(\frac{\phi'}{a}\right)^{\cdot}} \quad \frac{\phi''/a - \phi'\dot{a}}{a^2} = \frac{\phi''}{a^2} - \frac{\phi' H}{a}$$

$$\Rightarrow \frac{\phi''}{a^2} + 2H\frac{\phi'}{a} + \frac{\partial V}{\partial \phi} = 0 \rightarrow \boxed{\phi'' + 2aH\phi' + a^2\frac{\partial V}{\partial \phi} = 0}$$



→ A scalar field slowly rolling down a potential $V(\phi)$

→ The PE of such a field is very close constant so it quickly comes to dominat over KE

→ Inflation ends when ϕ reaches a value s.t. $V(\phi)$ is min & field will oscillate & decay into lighter particle

Slow Roll Models

↳ Hubble rate & zeroth order field vary slowly.

$$\rightarrow \eta = \underbrace{\int_{a_e}^{a} \frac{da}{Ha^2} \simeq \frac{1}{H}\int_{a_e}^{a}\frac{da}{a^2}}_{H \text{ is almost const.}} \simeq \boxed{-\frac{1}{aH}} \quad (7.16)$$

↳ $a_e >> a$ → scale factor before or in middle of inflation

(scale factor at the end of inflation)

→ Slow roll parameters → (vanish in the limit $\phi \to$ constant)

(i) $\boxed{\epsilon_{sr} = \frac{d}{dt}\left(\frac{1}{H}\right) = \frac{-H'}{aH^2}}$ $\left(= -\frac{\dot{H}}{H^2} = \frac{\frac{4\pi G}{3}\left(\rho + \frac{3P}{}\right) + 1}{H^2}\right)$

$\left(\frac{\dot{H}}{H^2} = \frac{a\ddot{a} - (\dot{a})^2}{(a)^2}\right)$

$\dot{H}$ is always (−ve)

Hence ϵ_{sr} is always +ve

$\frac{4\pi G}{3}(\rho + 3P) + \frac{8\pi G\rho}{3}$

$\frac{4\pi G}{3}(2\rho + 3P)$

→ In an acc. expansion $\dot{a}\uparrow, a\uparrow$

but $\ddot{a} << \dot{a}$ Hence $\dot{H}$ decreases slowly (Inflation era $\epsilon_{sr} \lesssim 1$)

→ In a deacc. expansion $\dot{a}\downarrow$ $a\uparrow$

hence H decreases rapidly. → (Radiation era $\epsilon_{sr} \sim 2$)

(ii) $\boxed{\delta_{sr}} = \frac{1}{H}\frac{\ddot{\phi}}{\dot{\phi}} = +\frac{1}{aH\phi'}\left[\phi'' - aH\phi'\right] = \boxed{-\left[3aH\phi' + a^2\frac{\partial V}{\partial \phi}\right] \cdot \frac{1}{aH\phi'}}$

$= \frac{1}{H}\cancel{\left[\frac{\phi''/a - \phi'\dot{a}}{a\,\phi^2}\right]} = \frac{1}{H}\left[\frac{\frac{\phi''}{a^2} - \frac{\phi' H}{a}}{\phi'/a}\right] = \frac{1}{H\phi'}\left[\frac{\phi''}{a} - \phi' H\right]$

↓

Nice intutive discussion after this on pg. 167. to c

7.3 Gravitational Wave Production

→ Scalar perturbations to the metric couple to the density of matter (i.e. $G^{0(1)}_{0} \neq 0$ for scalar perturb) & produce large scale structure.

→ Tensor perturb. have $G^{0(1)}_{0} = 0$ (6.4.4) & are not responsible for the large scale structure of universe.

→ Inflation generates both scalar & tensor fluctuations in the metric. Tensor fluctuations (grav. waves) induce anisotropies in the CMB.

→ Tensor perturbations are also gauge invariant. (while scalar perturb. are not)

→ Two paras above 7.3.1 (Great discussion)

7.3.1 Quantizing Harmonic Oscillator

→ A harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \left(\text{equiv. } E = \frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}\right)$$

→ In Heisenberg's picture (states are fixed but operators evolve) x is an operator given by

$$\hat{x} = v(\omega,t)\,\underset{\substack{\uparrow\\ \text{annihilation}}}{\hat{a}} + v^*(\omega,t)\,\underset{\substack{\uparrow\\ \text{creation operator}}}{\hat{a}^{+}}$$

(i) $\boxed{v(\omega,t) = \dfrac{e^{-i\omega t}}{\sqrt{2\omega}}}$ (ii) $[\hat{a}, \hat{a}^{+}] = [\hat{a}^{+}, \hat{a}] = 1$

(iii) $[\hat{x}, \hat{p}] = i$

→ Using this we calculate quantum fluctuations of operator $\hat{x}$ (Average of the square of fluctuations) in the ground state

$$\langle \hat{n}^\dagger \hat{n} \rangle_{|0\rangle} = \langle |\hat{n}|^2 \rangle_{|0\rangle} = \langle 0|\hat{n}^\dagger \hat{n}|0\rangle$$

$$= \langle \hat{n}\, 0 | \hat{n}\, 0 \rangle$$

$$= \langle 0|(v^* \hat{a}^\dagger + v\hat{a})(v\hat{a} + v^* \hat{a}^\dagger)|0\rangle$$

$$= \langle 0 | \overset{|v|^2}{\hat{a}^\dagger \hat{a}} + v^2 (\hat{a})^2 + (v^*)^2 (\hat{a}^\dagger)^2 + \overset{|v|^2}{\hat{a}\hat{a}^\dagger} |0\rangle$$

~~$= \langle 0|\hat{a}^\dagger \hat{a}|0\rangle + \langle 0|\hat{a}\hat{a}^\dagger|0\rangle +$~~

Note: $\hat{a}|0\rangle = 0$
↳ annihilates

$\underline{\langle 0|\hat{a}^\dagger = 0}$ → Taking ~~conjugates~~ duals

$$\Rightarrow |v|^2 \langle 0|\hat{a}\, a^\dagger|0\rangle = |v|^2 \langle 0|a^\dagger \hat{a} + [\hat{a}, \hat{a}^\dagger]|0\rangle$$

$$= |v|^2 \langle 0|\hat{N} + I|0\rangle$$

$$\boxed{\langle |\hat{n}|^2 \rangle = |v|^2} = \underline{1/2\omega} \quad - (7.26)$$

↓

We'll later identify $\hat{n}$ with field φ.

→ Note that we can visualise states $|1\rangle, |2\rangle$ as being the particles in vaccum $|0\rangle$.

→ While dealing with perturbations, we'll deal with an infinite collection of oscillators, one for every fourier mode '$\vec{k}$'.

→ In minkowski space the vaccum expectation (or variance) value is independent of time (eq 7.26) but this changes in a rapidly expanding space time.

→ The vaccum state $|0\rangle$ evolves during inflation & produce particles (gravitons) that form gravitational waves. The variance of the fluctuations will be identified as power spectrum of grav. waves.

7.3.2 Tensor Perturbations

$\rightarrow$ $$h'' + \frac{2a'}{a} h' + k^2 h = 0 \qquad (h = h_+, h_\times) \quad - (7.27)$$

$\downarrow$

Convert this in the form of harmonic oscillator so that h can be easily quantized.

Consider $$\mathfrak{h} = \frac{ah}{\underbrace{\sqrt{16\pi G}}}$$

Reason for this explained in book

$$\frac{h'}{\sqrt{16\pi G}} = \frac{\mathfrak{h}'}{a} - \frac{\mathfrak{h} a'}{a^2}$$

$$\frac{h''}{\sqrt{16\pi G}} = \frac{\mathfrak{h}''}{\underset{\sim}{a}} - \frac{2\mathfrak{h}' a'}{a^2} - \frac{\mathfrak{h} a''}{\underset{\sim}{a^2}} + \frac{2\mathfrak{h} (a')^2}{a^3}$$

not cancelled

from (7.27)

$$\left(\frac{\mathfrak{h}''}{a} - \cancel{\frac{2\mathfrak{h}' a'}{a^2}} - \frac{\mathfrak{h} a''}{a^2} + \cancel{\frac{2\mathfrak{h} (a')^2}{a^3}}\right) + \frac{2a'}{a}\left(\cancel{\frac{\mathfrak{h}'}{a}} - \cancel{\frac{\mathfrak{h} a'}{a^2}}\right) + \frac{k^2 \mathfrak{h}}{a} = 0$$

$$\frac{1}{a}\left[\mathfrak{h}'' + \left(k^2 - \frac{a''}{a}\right)\mathfrak{h}\right] = 0 \quad - (7.31)$$

$\hookrightarrow$ similar in form to quantum harmonic oscillator

We expect the sol^n. to be of the form

$$\hat{\mathfrak{h}}(\vec{k}, \eta) = v(k, \eta)\hat{a}_{\vec{k}} + v^*(k, \eta)\hat{a}^+_{\vec{k}}$$

where the coeffs are the roots of

$$\boxed{v'' + \left(k^2 - \frac{a''}{a}\right)v = 0} \quad - (7.33)$$

→ Before solving the diffⁿ eq 7.33 we'll first see how the eventual solutions determines the power spectrum of fluctuations.

Variance of perturbations in h field

$$\langle \hat{h}^{+}(\vec{k},\eta)\, \hat{h}(\vec{k}',\eta)\rangle_{|0\rangle} = \langle 0|\, \hat{h}^{+}(\vec{k},\eta)\,\hat{h}(\vec{k}',\eta)\,|0\rangle$$

$$= \langle 0|\,(v^{*}(\vec{k},\eta)\,\hat{a}^{+}_{\vec{k}} + v(\vec{k},\eta)\hat{a}_{\vec{k}})\,(v(\vec{k}',\eta)\hat{a}_{\vec{k}'} + v^{*}(\vec{k}',\eta)\hat{a}^{+}_{\vec{k}'})\,|0\rangle$$

$$= (v(\vec{k},\eta)\cdot v^{*}(\vec{k}',\eta))\ \underbrace{\langle 0|\,\hat{a}_{k}\,\hat{a}^{+}_{\vec{k}'}\,|0\rangle}$$

$$= (v(\vec{k},\eta)\cdot v^{+}(\vec{k}',\eta))\ \langle 0|\,\hat{N} + \underbrace{[\hat{a}_{k},\hat{a}^{+}_{k'}]}_{(2\pi)^3\,\delta^3(\vec{k}-\vec{k}')}\,|0\rangle \quad \text{(in 3-D space)}$$

??

$$\boxed{\langle\ \ \rangle = |v(\vec{k},\eta)|^2\,(2\pi)^3\,\delta^3_D(\vec{k}-\vec{k}')} \quad -(7.34)$$

↳ Vaccum expectation value of an operator $\hat{h}$, will be later identified as ensemble avg. of classical field.

→ A Quantum field is defined in all space, so it can be considered as an infinite collection of oscillators each at a different spacial position or in Fourier space at different values of $\vec{k}$. The quantum fluctuations in each of these oscillators are independent so $\hat{h}(\vec{k})$ is completely uncorrelated with $\hat{h}(\vec{k}')$ if $\vec{k} \neq \vec{k}'$.

We have $h(\vec{k},\eta) = \dfrac{ah}{\sqrt{16\pi G}}$

⇒ (from 7.34)

$$\boxed{\langle \hat{h}^{+}(\vec{k},\eta),\, \hat{h}(\vec{k}',\eta)\rangle = \frac{16\pi G}{a^2}\,|v(k,\eta)|^2\,(2\pi)^3\,\delta^3_D(\vec{k}-\vec{k}')}$$

$$= P_h(k,\eta)\,(2\pi)^3\,\delta^3_D(\vec{k}-\vec{k}')$$

Power spectrum of primordial tensor perturbations

↳ (kind of a measure of amplitude or (amplitude)² of a wave of a particular wave vector $\vec{k}$)

Can also define dimensionless power spectrum $\boxed{\Delta^2_h(k,\eta) = \dfrac{k^3}{2\pi^2}\,P_h(k,\eta)}$

$\rightarrow P_h(k,\eta) = 16\pi G \dfrac{|v(k,\eta)|^2}{a^2}$

↳ We've reduced the problem of determining the spectrum of tensor perturbations produced during inflation to one of solving a second order diffⁿ eqⁿ for $v(k,\eta)$.

→ Now we try to solve (7.33)
We first calculate $\dfrac{a''}{a}$ during inflation

$$a' = \frac{da}{d\eta} = \frac{da\, a}{a\,dt} = a^2 H \simeq \frac{-a}{\eta} \quad \text{(from 7.16)}$$

$$a'' = -\frac{a'}{\eta} + \frac{a}{\eta^2} \simeq \frac{2a}{\eta^2} \qquad \boxed{\frac{a''}{a} \simeq \frac{2}{\eta^2}}$$

$\Rightarrow$ (7.33 becomes) $v'' + \left(k^2 - \dfrac{2}{\eta^2}\right)v = 0$

Exercise (7.12)

Consider $\bar{v} = \dfrac{v}{\eta}$ $\quad \cancel{\bar{v}' = \dfrac{v'}{\eta} - \dfrac{v}{\eta^2}}$ $\quad \bar{v}'' = \cancel{\dfrac{v''}{\eta} - \dfrac{2v'}{\eta^2} + \dfrac{2v}{\eta^3}}$

$v = \eta\bar{v}$ $\quad v' = \bar{v} + \eta\bar{v}'$ $\quad v'' = 2\bar{v}' + \eta\bar{v}''$

$\Rightarrow (2\bar{v}' + \eta\bar{v}'') + \left(k^2 - \dfrac{2}{\eta^2}\right)\eta\bar{v} = 0$

$\cancel{\eta\bar{v}'' - 2}$ $\bar{v}'' + \dfrac{2\bar{v}'}{\eta} - \dfrac{2\bar{v}}{\eta^2} = -k^2\bar{v}$ → Bessel's Eqⁿ

Can check $\left(\dfrac{e^{-ik\eta}}{\eta}, \dfrac{ie^{-ik\eta}}{k\eta^2}\right.$ satisfy the eqⁿ)

→ The solⁿ $\boxed{v(k,\eta) = \dfrac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \dfrac{i}{k\eta}\right]}$

Note: Rel^n b/w $k|\eta|$ & aH

Perturbations of order k are considered for ~~outside~~ inside horizon when $\underbrace{(k^{-1})}_{\text{Perturbation length scale}} << \underbrace{(aH)^{-1}}_{\text{Horizon length scale}}$.

Also from 7.38 we have $aH \sim -\frac{\alpha}{\eta} \Rightarrow aH \sim -\frac{1}{\eta}$ (During inflation)

$\Rightarrow (k^{-1}) << (aH)^{-1} \simeq -(\eta)$

$\frac{1}{k|\eta|} << \frac{-\eta}{|\eta|}$

($\eta < 0$ before ~~horizon~~ inflation) $\rightarrow |k\eta| >> 1$ ~~bef~~ $\boxed{|k\eta| >> 1}$

$\Rightarrow$ Perturbations are far inside horizon for sufficiently many e-folds

→ After inflation has worked for sufficiently so that $(k^{-1}) >> (aH)^{-1})$ $((aH)^{-1}$ has decreased sufficiently $(\sim 0.)$ The mode has exited the horizon
$k|\eta|$ becomes very small

$$\lim_{-k\eta \to 0} v(k,\eta) = -\frac{e^{-ik\eta}\, i}{\sqrt{2k}\, k\eta}$$

(actually $\eta \to 0^-$)

→ Note that $P_h(k,\eta) \propto \frac{|v(k,\eta)|^2}{a^2}$. At early times (when k is well inside horizon) $k|\eta| >> 1$, hence $v(k,\eta) \sim \frac{e^{-ik\eta}}{\sqrt{2k}}$.

So amplitude scales as $(\sqrt{P_h})$ $1/a$ i.e. inflation reduces the amplitude of modes.

→ As inflation reduces hubble horizon $(aH)^{-1}$, mode k eventually exits the horizon after which amplitude $\propto \sqrt{P_h} \propto \frac{|v(k,\eta)|}{a}$ $\propto \frac{1}{|a\eta|}$ & which is a const. (as $\eta \propto \frac{1}{a}$).

→ This mode k becomes an observable gravitational wave once k re-enters the horizon. (12)

from (7.16)

$$\rightarrow P_h(k) = \frac{16\pi G}{a^2}\frac{1}{2k^2\eta^2} = \frac{16\pi G}{a^2}\frac{1}{2k^3}(a^2H^2) = \boxed{\frac{\cancel{16}8\pi G H^2}{\cancel{2}k^3}}$$

→ In deriving ~~that~~ $\eta = \frac{1}{aH}$ we've assumed H is constant which is the case only during inflation (there also it varies slowly), the result remains accurate when mode of interest leaves the horizon $k|\eta| = 1$.

→ Nice discussion above two paras of section 7.4.

7.4 Scalar Perturbations

→ Inflation theory predicts adiabatic perturbations: ~~to~~ different patches ~~have~~ of the universe have different overdensities, but the fractional density perturbations are the same for all species.

$$\boxed{\frac{\delta\rho_s}{\delta\rho} = \frac{\delta\rho}{\rho}}$$

→ D (i) How is this related to ~~adia~~ conventional adiabatic processes ??

(ii) How is this related to the definition in Baumann's lectures ??

(i.e. perturbations are just time shifted values of density at diff. posn's)

7.4.1 Scalar field perturbations around an unperturbed background

$$\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t)$$

zeroth

→ First we derive evolution of $\delta\phi$ in ~~an~~ smooth expanding universe ~~& then~~ (i.e. metric $g_{00} = -1$ & $g_{ij} = \delta_{ij}a^2(\eta)$) & then in sec 7.4.2 & 7.4.3 consider the perturbations.

→ Using ~~tens~~ stress energy tensor conservation eq^n we get $\nabla_\mu T^\mu{}_\nu = 0$ & considering $\nu = 0$ comp. we get (eq^n 7.11)

$$\nabla_\mu T^\mu{}_0 = \frac{\partial T^\mu_0}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu}T^\alpha_0 - \Gamma^\alpha_{0\mu}T^\mu_\alpha$$

(~~Note $T^\alpha_\beta = 0$ $\forall \alpha \neq \beta$ since $g_{\mu\nu} = 0$ $\forall \mu \neq \nu$~~)

Using christoffel symbols ~~for~~ from (2.24 - 2.25)

$(\Gamma^0{}_{ij} = \delta_{ij}\,\dot a\, a \,,\ \Gamma^i{}_{0j} = \delta_{ij}\frac{\dot a}{a})$ rest all 0

$$0 = \nabla_\mu T^\mu{}_0 = \frac{\partial T^0_0}{\partial t} + 3\frac{\dot a}{a} T^0{}_0 - \frac{\dot a}{a} T^i{}_i + \underbrace{\frac{\partial T^i_0}{\partial x^i}}_{ik_i T^i_0}$$

Considering the eq^n $\nabla_\mu T^\mu_0$ to first order we get

$$\boxed{\frac{\partial}{\partial t}\delta T^0_0 + ik_i\delta T^i_0 + 3H\delta T^0_0 - H\delta T^i_i = 0} \quad -(7.45)$$

Now we compute δT^μ_ν terms in terms of perturb to scalar field $\delta\varphi$ using 7.6

$$\rightarrow \delta T^i{}_0 = +g^{i\nu}\frac{\partial\varphi}{\partial x^\nu}\frac{\partial\varphi}{\partial x^0} = +g^{ii}\underbrace{\frac{\partial\varphi}{\partial x^i}}_{\text{first order}}\underbrace{\frac{\partial\varphi}{\partial x^0}}_{\text{zeroth order considered}} = +\frac{1}{a^2}ik_i\delta\varphi\frac{\bar\Phi'}{a}$$

$$\boxed{\delta T^i{}_0 = \frac{ik_i\bar\varphi'\delta\varphi}{a^3}}$$

$$\rightarrow \text{similarly}\quad \delta T^0_0 = 2(-1)\frac{\bar\Phi'(\delta\varphi)'}{a^2} - (-1)\frac{\bar\Phi'}{a}\frac{(\delta\varphi)'}{a} - \underbrace{V(\bar\Phi+\delta\varphi)}_{\left(V(\bar\Phi) + \frac{\partial V}{\partial\varphi}\delta\varphi\right)}$$

$$\boxed{\delta T^0_0 = -\frac{\bar\varphi'\delta\varphi'}{a^2} - \frac{\partial V}{\partial\varphi}\delta\varphi}$$

$$\rightarrow \delta T^i{}_j = -\delta^i{}_j\left[(-1)\frac{\bar\Phi'}{a}\frac{(\delta\varphi)'}{a} + V(\bar\Phi+\delta\varphi)\right]$$

$$\boxed{\delta T^i{}_j = \delta_{ij}\left[\frac{\bar\varphi'\delta\varphi'}{a^2} - \frac{\partial V}{\partial\varphi}\delta\varphi\right]}$$

$\rightarrow$ eq^n (7.45) becomes

$$+\left(\frac{1}{a}\frac{\partial}{\partial \eta} + 3H\right)\delta T^0_0 - \frac{k^2}{a^3}\bar\varphi'\delta\varphi - 3H\left[\frac{\bar\varphi'\delta\varphi'}{a^2} - \frac{\partial V}{\partial\varphi}\delta\varphi\right] = 0$$

$$+\left[\frac{2\bar\varphi'\delta\varphi'}{a^4}a' - \frac{\bar\varphi''\delta\varphi'}{a^3} - \frac{\bar\varphi'\delta\varphi''}{a^3} - \cancel{\frac{3\bar\varphi'\delta\varphi' a'}{a^4}} - \frac{1}{a}\frac{\partial^2 V}{\partial^2\varphi}\bar\varphi'\delta\varphi - \frac{1}{a}\frac{\partial V}{\partial\varphi}(\delta\varphi)' - \cancel{\frac{3a'}{a^2}\frac{\partial V}{\partial\varphi}\delta\varphi}\right]$$

$$- \frac{k^2}{a^3}\bar\varphi'\delta\varphi - \underbrace{\cancel{\frac{3\bar\varphi'\delta\varphi' a'}{a^4}}}_{\text{(not cancelled)}} + \cancel{\frac{3a'}{a^2}\frac{\partial V}{\partial\varphi}\delta\varphi}$$

Multiplying by a^3 we get

$$-\underbrace{\frac{\bar{\Phi}'' \delta\varphi'}{a^3} \mp a^2 \frac{\partial V}{\partial \varphi} \delta\varphi' - 4aH\bar{\Phi}'\delta\varphi'}_{\downarrow} \mp \bar{\Phi}'\delta\varphi'' \pm a^2 \frac{\partial^2 V}{\partial \varphi^2} \bar{a}'\delta\varphi$$

$$- k^2 a \bar{\omega}' \delta\varphi = 0$$

- Using eq^n (7.15) we get

$$-\delta\varphi' \, 2aH\bar{\Phi}' - \bar{\Phi}'\delta\varphi'' - k^2 a \, \Phi'\delta\varphi + \underbrace{a^2 \frac{\partial^2 V}{\partial \varphi^2}}\bar{\varphi}'\delta\varphi = 0$$

(Q) ~~How~~ Is it that small that we can neglect it in comparison to perturbation. We prove that $\frac{\partial^2 V}{\partial \varphi^2}$ is typically small of the order of slow roll variables ϵ_{sr} & δ_{sr} so it can be neglected.

Proof: First we use the results from Eq (7.7 (a) & (b)) & we work under the assumptions (slow roll field)

~~$\epsilon_{sr} = \frac{d}{dt}\left(\frac{1}{H}\right)$~~ = (i) $\ddot{\varphi} \sim 0$ (φ is a slow roll field) [Most of the energy of this field is concentrated in potential form]

(ii) $H \sim \frac{8\pi G}{3} V(\varphi)$

Eq 7.8 (i) $\epsilon_{sr} = \frac{4\pi G(\dot{\varphi})^2}{H^2}$ (from 7.114)

$$= \frac{4\pi G}{\frac{8\pi G V(\varphi)}{3}} \left[\frac{\partial V/\partial\varphi}{3H}\right]^2 \quad \text{(7.14 \& using } \ddot{\varphi}=0)$$

$$= \frac{4\pi G}{\frac{8\pi G \times 8\pi G \times 9}{3 \times 3}} \left[\frac{\partial V/\partial\varphi}{V(\varphi)}\right]^2$$

$$\boxed{\epsilon_{sr} = \frac{1}{16\pi G}\left[\frac{\partial V/\partial\varphi}{V(\varphi)}\right]^2}$$

(ii) $\delta_{sr} = -\frac{1}{H}\frac{\ddot{\varphi}}{\dot{\varphi}}$; $\dot{H} = -\frac{1}{2}\dot{\varphi}^2(8\pi G)$ (2nd Friedmann eq^n)

$$\ddot{H} = -\dot{\varphi}\ddot{\varphi}\,(8\pi G)$$

$$\delta_{sr} = \frac{\ddot{H}}{(8\pi G) H (\dot{\varphi})^2} = \frac{\ddot{H}}{H}$$

→ We're finally left with

$$\boxed{\delta\Phi'' + 2aH\delta\Phi' + k^2\delta\Phi = 0}$$ → similar to the eq^n for tensor perturbation.

→ Sol^{n}s are similar to that of tensor eq^n & power spectrum as well.

$$\boxed{P_{\delta\Phi} = \frac{H^2}{2k^3}}$$ → (factor of $16\pi G$ missing for which justification is in Dodelson pg. 176) → (7.55)

→ By neglecting $\frac{\partial^2 V}{\partial \varphi^2}$ we have essentially set mass of inflaton to zero, so $\delta\Phi$ obeys the eq^n of massless field in an expanding universe just like massless graviton.

7.4.2 Super Horizon Perturbations

Note: From eq (5.48) we have

$$k^2(\Phi + \Psi) = -32\pi G a^2 [\rho_r \Theta_2 + \rho_\nu N_2] = \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^i_j\right) T^i_j \quad (\delta T^i_j)$$

for scalar field we have

$$k^2(\Phi + \Psi) = \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^i_j\right) \delta^i_j (\ ) = \left(\rho(\hat{k})^2 - \frac{3}{3}\right)(\ ) = 0$$

$\Rightarrow$ if ~~perturbations to~~ T^i_j is diagonal

$$\boxed{\Phi = -\Psi}$$

→ We'll start considering metric perturbations $\Phi, \Psi (= -\Phi)$ for a diagonal stress energy tensor.

→ We'll ~~prove~~ show that when the wavelength of the perturbation is much smaller than the horizon, we can in fact neglect metric perturbation.

→ We first write the eq^n of conservation of stress energy tensor, this time in presence of metric perturbations.

$$\frac{\partial}{\partial t}\delta T^0_0 + ik_i \delta T^i_0 + 3H\delta T^0_0 - H\delta T^i_i + 3\underbrace{(\rho + P)}_{\text{zeroth order } \rho \,\&\, P.}\dot{\phi} = 0 \quad -(7.56)$$

↳ We'll verify the fact that $\phi \sim \delta T^0_0/\rho$. This means that all terms (except last one) in 7.56 are of order $\sim \rho\Psi$. while $|\rho + P| << \rho$ during inflation. Hence the last term $(1/2\dot{\phi}^2)$ $(\frac{1}{2}\dot{\phi}^2 + V(\phi))$ is negligible during inflation.

→ ~~As in~~ The inequality $|\rho + P| << \rho$ would no longer holds as inflation ~~starts to~~ nears termination & the term $(\rho+P)\dot{\Psi}$ would start appearing in eq^{n}s. Physically this means that at some point we need to convert perturbations in the scalar field ($\delta\phi$) (which decay into standard Model particles) into those in the gravitational potential.

→ One way to deal with the coupling b/w the metric perturbation & those of energy density is to define the curvature perturbation $\mathcal{R}$.

$$\mathcal{R}(\vec{k},\eta) = \frac{i k_i \, \delta T^i{}_0(\vec{k},\eta)\, a^2 H(\eta)}{k^2[\rho+P](\eta)} - \Psi(\vec{k},\eta)$$

(i) $\rho+P(\eta) = \dot{\phi}^2 = \dfrac{(\phi')^2}{a^2}$ (ii) $i k_i \delta T^i{}_0 = -\dfrac{k^2 \phi' \delta\phi}{a^3}$ (iii) $\Psi \sim 0$ during inflation

$$\Rightarrow \quad \mathcal{R} = \frac{-k^2 \phi' \delta\phi\, a^2 H(\eta)}{k^2 a^3 \frac{(\phi')^2}{a^2}} = \boxed{-\frac{a H \delta\phi}{\phi'}} \quad \text{(during inflation)}$$

→ After inflation ends & the universe enters the radiation dominated era we have

$$T^0{}_i(\vec{n},t)_{rad.} = g\, a (1+\Phi-\Psi) \int \frac{d^3p}{(2\pi)^3} p_i f$$

$$= g a (1+\Phi-\Psi) \int \frac{d^3p}{(2\pi)^3} p_i \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p}(p,t)\, \Theta(\vec{n},\hat{p},t) \right]$$

$$\delta T^0_i(\vec{n},t) = \cancel{g a (\Phi-\Psi) \int \frac{d^3p}{(2\pi)^3} p_i f^{(0)}} + g a \int \frac{d^3p}{(2\pi)^3} p_i \left(-p\frac{\partial f^{(0)}}{\partial p}(p,t)\, \Theta(\vec{n},\hat{p},t)\right)$$

p_i: $p \sin\theta\cos\varphi$, $p\sin\theta\sin\varphi$, $p\cos\theta$ → 0 $\qquad \left\{ \mu \equiv \frac{\vec{k}\cdot\hat{p}}{k} \right\}$

$$i k_i \delta T^0{}_i(\vec{n},t) = i g a \int \frac{d^3p}{(2\pi)^3} k p \mu \left(-p \frac{\partial f^{(0)}}{\partial p}(p,t)\right) \Theta(\vec{n},\hat{p},t)$$

$$= g a k i \int \frac{-p^4}{(2\pi)^3} \frac{\partial f^{(0)}}{\partial p} dp \int \underbrace{d\Omega}_{2\pi\, d\mu} \mu\, \Theta(\vec{n},\hat{p},t)$$

$$= g a k i \int \frac{-p^4}{(2\pi)^3}\frac{\partial f^{(0)}}{\partial p} dp \int 2\pi\, \mu\, \Theta(\vec{n},\hat{p},t)\, d\mu$$

$$= -g a k\, i\, i\, \Theta_1 \underbrace{\int 4 \frac{p f^{(0)} d^3p}{(2\pi)^3}}_{\rho}$$

$$\boxed{i k_i \delta T^i{}_0 = g^{ii} g_{00} (i k_i \delta T^0_i) = -\frac{4 g a k \rho_r \Theta_1}{a}} \rightsquigarrow \Theta_1 \text{ (~~In Dodelson a is in~~ } d\tau \text{ (?))}$$

→ Also for rad' $P = \frac{\rho}{3}$

$$\Rightarrow \mathcal{R} = \frac{4 g a k \rho_r \Theta a^2 H}{k^2 \frac{4\rho}{3}} - \Psi = -\frac{3 a^3 H \Theta_1 g}{k} - \Psi = \underbrace{-\frac{3\Psi}{2}}_{\text{(in section 7.5)}}$$

→ From fig. 7.6 in dodelson, we see that $\mathcal{R}$ is const. (conserved) when the perturbations move outside horizon (will be shown later how $\mathcal{R}$ is conserved.

$\mathcal{R}$ (during inflation {horizon crossing}) = $\mathcal{R}$ (post inflation)

$$\cancel{\frac{-3\Psi}{2}\Big|_{\text{post inf}} = -aH\frac{\delta\phi}{\bar{\phi}'}}$$

$$\cancel{\Psi = \frac{2}{3}aH\frac{\delta\phi}{\bar{\phi}'}}$$

$$\boxed{-aH\frac{\delta\phi}{\bar{\phi}'}\Big|_{\substack{\text{horizon}\\\text{crossing}}} = \frac{-3\Psi}{2}\Big|_{\text{post inflation}}}$$

$$\boxed{\Psi\Big|_{\substack{\text{post}\\\text{inflation}}} = \frac{2}{3}aH\frac{\delta\phi}{\bar{\phi}'}\Big|_{\substack{\text{horizon}\\\text{crossing}}}}$$

Note that horizon crossing happens well before termination of inflation

In terms of power spectrum

$$P_\Psi(k)\Big|_{\text{post inflation}} = \frac{4}{9}\left(\frac{aH}{\bar{\phi}'}\right)^2 P_{\delta\phi}(k)\Big|_{aH=k}$$

$$\boxed{P_\Psi(k) = \frac{4}{9}\left(\frac{aH}{\bar{\phi}'}\right)^2 \times \frac{H^2}{2k^3} = \cancel{\frac{4}{9}} = \frac{2}{9k^3}\left(\frac{aH^2}{\bar{\phi}'}\right)^2\Big|_{aH=k}}$$

From exercise (7.7 (b)) $(aH/\bar{\phi}')^2 = 4\pi G/\epsilon_{sr}$

$$\boxed{\underbrace{P_\Psi(k)}_{} = P_\Phi(k) = \frac{8\pi G}{9k^3}\frac{H^2}{\epsilon_{sr}}\Big|_{aH=k}} \quad -(7.62)$$

(from $\Psi = -\Phi$)

(From 7.42 we get that $P_h(k) \sim \epsilon_{sr} P_\Psi(k)$)
(Hence (as expected) scalar perturbations dominate as comp. to tensor mode perturbations)

$$\boxed{P_\Phi(k) = P_\Psi(k) = \frac{128\pi^2 G^2}{9k^3}\left(\frac{H\,V(\phi)}{\partial V/\partial\phi}\right)^2\Big|_{aH=k}} \quad -(7.63)$$

(from exercise 7.8 (a))
$\left(\epsilon_{sr} = \frac{1}{16\pi G}\left(\frac{\partial V/\partial\phi}{V}\right)^2\right.$

→ A nice ~~physical~~ discussion about the physical interpretation of the result in 7.63 is give in Dodelson.

→ We now prove that $\mathcal{R}$ is conserved on superhorizon scales. (18)

→ We return to eqⁿ (7.56) (conservation eqⁿ) for this

→ From ex. 7.13 we see that $k_i\, \delta T^i_0$ is prop. to k^2 on large scales & hence it can be neglected, so we have

↳ Note that this δT^i_0 is diff. from the one we calc. on pg16 as it is calculated during inflation & it is generated by (super-horizon) metric perturb.

$$\frac{\partial}{\partial t}\delta T^0_0 + 3H\,\delta T^0_0 - H\,\delta T^i_i = -3(\rho+P)\dot{\Psi}$$

Using result of exercise 7.13 we have

$$\mathcal{R} = \frac{i k_i\, \delta T^i_0 a^2 H}{k^2(\rho+P)} - \Psi = -\Psi - \frac{1}{3}\frac{\delta T^0_0}{\rho+P}$$

$$\frac{\partial \mathcal{R}}{\partial t} = -\dot{\Psi} - \frac{1}{3}\frac{\partial}{\partial t}\left(\frac{\delta T^0_0}{\rho+P}\right)$$

$$\frac{\partial}{\partial t}\delta T^0_0 + 3H\,\delta T^0_0 - H\,\delta T^i_i = 3(\rho+P)\left[\frac{\partial \mathcal{R}}{\partial t} + \frac{1}{3}\frac{\partial}{\partial t}\left(\frac{\delta T^0_0}{\rho+P}\right)\right]$$

$$\delta T^0_0\left[3H + \frac{1}{(\rho+P)}\left(\frac{d\rho}{dt} + \frac{dP}{dt}\right)\right] - H\,\delta T^i_i = 3(\rho+P)\frac{\partial \mathcal{R}}{\partial t}$$

From eqⁿ (2.56) we have $\frac{d\rho}{dt} = -3H(\rho+P)$

$$\text{LHS} = \frac{\delta T^0_0}{(\rho+P)}\frac{dP}{dt} - H\,\delta T^i_i$$

→ $\left[T^0_0 = -\rho \quad \delta T^0_0 = -\delta\rho \right]$

$$= 3H\,\delta T^0_0 \frac{\dot{P}}{\dot{\rho}} - H\,\delta T^i_i = 3H\left[\frac{\dot{P}}{\dot{\rho}}\delta\rho - \delta P\right]$$

$$\boxed{3H\left[\frac{\dot{P}}{\dot{\rho}}\delta\rho - \delta P\right] = 3(\rho+P)\frac{\partial \mathcal{R}}{\partial t}} \quad - (7.70)$$

→ $\frac{\partial \mathcal{R}}{\partial t} = 0$ if $\frac{\dot{P}}{\dot{\rho}}\delta\rho = \delta P$

→ At the background level P & ρ are functions of $\bar{\Phi}$ only so we can write

$$\delta P = \frac{dP}{d\bar{\varphi}}\delta\varphi \quad \& \quad \delta\rho = \frac{d\rho}{d\bar{\varphi}}\delta\varphi$$

$$\dot{P} = \frac{dP}{d\bar{\varphi}}\dot{\bar{\Phi}} \quad \& \quad \dot{\rho} = \frac{d\rho}{d\bar{\varphi}}\dot{\bar{\varphi}}$$

$$\Rightarrow \frac{\dot{P}}{\dot{\rho}} = \frac{\delta P}{\delta\rho} \qquad \text{hence } \frac{\partial \mathcal{R}}{\partial t} = 0$$ → (nice argument in Dodelson)

↳ ① This proves that $\frac{\partial \mathcal{R}}{\partial t} = 0$ in the region ~~when~~ between $k^{-1} \ll (aH)^{-1}$ & the end of inflation. Also in the region after the end of inflation this φ field has been dissociated into ~~scalar fie~~ particles which have an eqn of state some $P = k\rho^{\alpha}$ & which would also yield $\frac{\dot{P}}{\dot{\rho}}\delta\rho - \delta P$ & thus $\frac{\partial \mathcal{R}}{\partial t} = 0$ but how do we know what happens in intermediate region (just as inflation is ending) ?? (it might ~~be~~ a case $\mathcal{R}$ changes its value ~~there~~ at the inflation ending & settles at some other value ??)

7.4.3 Spatially Flat Slicing

→ The results of the previous section ($\mathcal{R}$ being constant & $P_{\Phi}(k)$) ~~exp~~ can be obtained in a much more elegant way using gauge-invariant variables.

→ In conformal Newtonian gauge the perturbations to the scalar field $\delta\phi$ are coupled to the potential Φ. (For example eq' (7.56))

→ We consider the gauge with "spatially flat slicing" such that the spatial part of the metric obeys $g_{ij} = a^2\delta_{ij}$. In this gauge the line element is

$$ds^2 = -\left[1 + 2A(\vec{x},t)\right]dt^2 - 2a(t)\frac{\partial B}{\partial x^i}(\vec{x},t)\,dx^i dt + a^2(t)\delta_{ij}\,dx^i dx^j$$

→ Two functions A, B which characterize the scalar perturbations.

→ We obtain the eqⁿ for $\delta\varphi$ in this gauge. (like 7.54)

For eqⁿ 7.45 we require $T^0{}_0$, $T^i{}_0$, $T^i{}_i$

$$T^0{}_0 = g^{0\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial t} - \frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - V(\phi)$$

$$\delta T^0{}_0 = \frac{-2\bar{\Phi}'\delta\varphi'}{a^2} + \frac{2A(\bar{\Phi}')^2}{a^2} - \frac{1}{2}g^{i\nu}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^\nu} - V(\varphi)$$

$$\boxed{\delta T^0{}_0 = \frac{-\bar{\Phi}'\delta\varphi'}{a^2} + \frac{A(\bar{\Phi}')^2}{a^2} - \cancel{\frac{1}{2}}\cancel{\frac{1}{a^2}} - V(\varphi)} = \frac{1}{2}g^{0\nu}\frac{\partial\varphi}{\partial x^0}\frac{\partial\varphi}{\partial x^\nu} - V(\varphi)$$

$$\delta T^i{}_0 = g^{i\nu}\frac{\partial\varphi}{\partial x^\nu}\frac{\partial\varphi}{\partial x^0}$$

$$\delta T^i{}_i = g^{i\nu}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^\nu} - \frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - V(\varphi)$$

$$= 3\left[-\frac{1}{2}g^{0\nu}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x^\nu} - V(\varphi)\right]$$

Putting in eqⁿ (7.45)

$$\frac{\partial}{\partial t}(\delta T^0_0) + 6H\underbrace{\frac{1}{2}g^{0\nu}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x^\nu}} = 0$$

→ In this gauge we can obtain the eqⁿ (7.54) exactly without ~~using any approximation~~ (neglecting any couplings (as A & B don't couple to φ in this gauge).

Recall that while deriving eqⁿ 7.54 we neglected Φ term in (7.56) during inflationary epoch ~~& thus~~

→ This is the advantage of working in the spatially flat slicing gauge.

→ Once we obtain 7.54 we can immediately write the power spectrum for $\delta\varphi$ using 7.55

→ Now we choose a gauge-invariant variable.

→ The choice of this variable is nicely explained in the Dodelson above eq 7.73.

→ We consider a Bardeen variable

~~$V(\vec{k},t)$~~

$$v(\vec{k},t) = B(\vec{k},t) + \frac{i k_i a\, \delta T^i_0 (\vec{k},t)}{k^2\; \rho + P}$$

→ In conformal Newtonian gauge it can be easily checked that this variable for matter is $\vec{U}_m = i\vec{k}\, v$ & for radiation $ikv = -3i\,\Theta_{r,i}$ (proportional to dipole)

→ In spatially flat gauge

$$\cancel{\delta T^i_a} = \delta T^i_0 = g^{i\nu} \frac{\partial\phi}{\partial x^\nu}\frac{\partial\varphi}{\partial x^0} = \frac{1}{a^2}\, i k_i\, \delta\varphi \frac{\bar{\Phi}'}{a}$$

↳ (D) Why don't we consider here $g^{i0}\frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x^0}$ (?)

→ $$\boxed{v = B - \frac{\bar{\Phi}'\,\delta\varphi}{(\rho+P)a^2}}$$

→ Φ_H defined in (6.19) is also a gauge independent quantity & is equal to aHB since $D = E = 0$ in the spatially flat gauge.

So the quantity $\mathcal{R} = -\Phi_H + aHv$ is also a gauge-invariant

→ Note that in conformal newtonian gauge $\Phi_H = -\phi = \varphi$

& → $v = 0 + \frac{i k_i a\,\delta T^i_0 (\vec{k},t)}{k^2(\rho+P)}$ Hence $\mathcal{R}$ defined in eqn (7.57)

is indeed equal to $\mathcal{R}$ defined as $\mathcal{R} = -\Phi_H + aHv$

→ In spatially flat slicing $\mathcal{R} = aH(v-B) = -\frac{aH\bar{\Phi}'\delta\phi}{\frac{\bar{\Phi}'^2}{a^2}\,\cancel{\frac{1}{2}\bar{\Phi}'^2}\, a^2} = \boxed{-\frac{aH}{\bar{\Phi}'}\delta\phi}$

→ $P_{\mathcal{R}}(\vec{k}) = \left(\frac{aH}{\bar{\phi}'}\right)^2 P_{\delta\phi}(k)$

→ $$\boxed{P_{\mathcal{R}}(k) = \frac{4\pi G}{\epsilon_{sr}} \times \frac{H^2}{2k^3}\Big|_{aH=k} = \frac{2\pi G H^2}{\epsilon_{sr} k^3}\Big|_{aH=k}}$$

~~Very~~ Power spectrum of a gauge invariant quantity

→ ~~In~~ We argued previously that post inflation $\mathcal{R} = \frac{3\Phi}{2}$ (in newtonian gauge)

⇒ $P_{\mathcal{R}} = \frac{9}{4} P_{\Phi}$ → $\boxed{P_{\Phi} = \frac{4}{9} P_{\mathcal{R}} = \frac{8\pi G H^2}{9k^3 \epsilon_{sr}}\Big|_{aH=k}}$ (as obtained from 7.62)

→ Φ_H has a ~~nice~~ geomet. interpretation that the curvature of three dimensional space at fixed time is equal to $\frac{4k^2 \Phi_H}{a^2}$.

→ Therefore perturbations in Φ_H represent ~~perturb in R~~ curvature perturbations. (even though the zeroth order piece is euclidean, perturbations induce a curvature that varies from place to place).

→ $\mathcal{R}$ is combination of both Φ_H & velocity (v). If we move to comoving gauge, velocity vanish & $\mathcal{R}$ is equal to Φ_H. In comoving gauges $\mathcal{R}$ corresponds to a curvature perturbation & scalar perturbations generated during inflation are often called curvature perturbations.

7.5 The Einstein - Boltzmann Eq's at early times

→ We'll consider first the Boltzmann eq"s at (5.67-5.73) at very early times. ~~In par~~ We consider (i.e. $\eta > 0$ but small): times so early that for any k-mode of interest $k\eta \ll 1$ ($k/aH \ll 1$) & hence no perturbations are inside the horizon.

→ Consider eqⁿ (5.67)

$$\Theta' + ik\mu\Theta = -\Phi' - ik\mu\Psi - \tau'\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$\Theta' \sim \frac{\Theta}{\eta}$ $\quad ik\mu\Theta \sim k\Theta$ → Since we are in the regime $k << \frac{1}{\eta}$ we can neglect all terms which have k in multiple.

~~Since~~ → Also all perturbations of interest have wavelengths $\sim k^{-1}$ much larger than the distance over which causal physical operates (the length of the horizon) hence we can neglect higher multipoles ($\Theta_1, \Theta_2 \ldots$) in comparison to Θ_0.

→ We can also neglect terms v_b as ~~those are also of~~ (Reason) (??)

~~the~~

→ For photons & neutrons we have

$$\boxed{\begin{aligned}\Theta_0' + \Phi' &= 0\\ N_0' + \Phi' &= 0\end{aligned}} \quad -(7.81)$$

→ Using similar principles for eqⁿ (5.69 & 5.71) for CDM & baryons we have

$$\boxed{\begin{aligned}\delta_c' &= -3\Phi'\\ \delta_B' &= -3\Phi'\end{aligned}} \quad -(7.82)$$

⇒ Outside the comoving horizon gravity is the only relevant force, this is the reason both ~~gravity &~~ dark matter & baryons follow same eqⁿ.

→ Now we consider Einstein's eqⁿ at earlier times.

From eqⁿ (6.41) we have $\quad \rho_r\Theta_{r,0} = \rho_\gamma\Theta_0 + \rho_\nu N_0$

$$3\left(\frac{a'}{a}\right)\left(\Phi' - \frac{a'}{a}\Psi\right) = 16\pi G a^2 \rho_r \Theta_{r,0}$$

↳ (Here we've neglected k^2 terms & matter density terms as we're in the radiation dominated era)

→ Radiation domination ⇒ $d\eta = \frac{dt}{a} = \frac{da}{a^2 H} \propto \frac{da}{a^2 \, 1/a^2} \propto da$ $\quad$ $\eta \propto a$. $\quad$ also $\frac{a'}{a} = aH = \frac{1}{\eta}$

(?) {In rad. dominated era H is not constant so this approx is not valid) (??)

$$\to \frac{\Phi'}{\eta} - \frac{\Psi}{\eta^2} = \frac{16\pi G \rho a^2 \Theta_{r,0}}{3} = \frac{2}{\eta^2}\Theta_{r,0} \quad \left\{H^2 = \frac{8\pi G\rho}{3}\right\} - (7.84)$$

(We can do this since both Θ_0 & N_0 follow similar eqⁿ initially)

$\rightarrow \quad \underline{\Phi}'\eta - \underline{\Psi} = 2\Theta_{r,0} \quad - (7.85)$

Diff^n both sides & using (7.81)

$\underline{\Phi}''\eta + \underline{\Phi}' - \underline{\Psi}' = -2\underline{\Phi}'$

From (~~7.68~~ 6.48) we see how the higher moments ~~$\Psi = \Phi$~~ or of photon & neutrino dist. source $\Psi + \Phi$. We neglect these higher order moments here & use $\underline{\Psi} = -\underline{\Phi}$. This yields

$\underline{\Phi}''\eta + 4\underline{\Phi}' = 0 \quad - (7.88)$

→ Using $\underline{\Phi} = \eta^p$ we can arrive at two sol^{n}s $p = 0, -3$. $p=-3$ mode is the decaying mode. $\underline{\Phi} = \eta^{-3}$ decays as $\eta \uparrow$. while $p=0$ does not decay if excited & is of interest to us. We consider this mode only from now on.

→ Eq^n 7.85 gives us $\underline{\underline{\Phi} = 2\Theta_{r,0}}$ $\Rightarrow$ ($\Theta_{r,0}$ as well as its const. $\Theta_0, \mathcal{N}_0$ remain const. in time) we have

→ For adiabatic perturbations ~~$\Theta_{r,0}$~~

$$\Theta_0(\vec{k}, \eta_i) = \mathcal{N}_0(\vec{k}, \eta_i)$$

$$\Rightarrow \quad \boxed{\underline{\Phi}(\vec{k}, \eta_i) = 2\Theta_0(\vec{k}, \eta_i)}$$

→ Combining (7.81 & 7.82) we have

$$\boxed{\delta_c(\vec{k}, \eta) = 3\Theta_0(\vec{k}, \eta) + \underbrace{C}_{\text{const.}}}$$

dark matter overdensity

↳ For baryon density we've similar eq^n with same const. C due to adiabatic perturb.

→ Now we prove that this const. C is also zero.

→ Since we are working in adiabatic perturbations regime the ~~no. density~~ perturbations in no. density of ~~both~~ all species must be same.

$$\frac{\delta n_c}{n_c} = \frac{\delta n_r}{n_r}$$

$n_c = \bar{n}_c(1 + \delta_c)$ & $n_r = \bar{n}_r(1 + 3\Theta_0)$ → (Can be calc. from $n_r = \int \frac{d^3p}{(2\pi)^3} f$, $f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$)

$$\Rightarrow \quad \boxed{\delta_c = 3\Theta_0} = \boxed{\delta_b}$$

→ We determine initial cond^n for velocities & dipole moments of matter & radiation respectively.
(this has been done in eq 7.17 a little bit handwavingly)

$$\Theta_1(\vec{k},\eta) = \mathcal{N}_1(\vec{k},\eta) = \frac{i v_b(\vec{k},\eta)}{3} = \frac{i v_c(\vec{k},\eta)}{3} = -\frac{k}{6aH}\Phi(k,\eta)$$

Putting in eq^n (7.59)

we get $\mathcal{R} = \dfrac{-3aH\Theta_1}{k} - \Psi = \boxed{\dfrac{-3\Phi}{2}}$