Black Holes and Information

Instructor: Prof Suvrat Raju
Lecture Notes

Prakhar Bansal
https://prakharbansal16.github.io

Last update: 2023-01-16 23:07:02Z

Contents

1 Lecture 5: QFT in Curved Spacetime

1.1 Euler Langrange Equations in Curved Background

- Deal with free fields coupled with bg curvature
- Due to accelerated observers, particles are observer dependent

$$S = \frac{1}{2}\int \sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2]d^4x \tag{1}$$

Note:

- ϕ is a minimally coupled scalar
- The metric g conisdered is a fixed metric

Deriving E-L equations

The langrangian density is given by

$$\mathcal{L} = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2.$$

E-L equations are given by

$$\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\phi} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\alpha\phi)}\right) \tag{2}$$

$$-2m^2\phi\sqrt{-g} = \frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\mu\nu}(\partial_\nu\phi)\delta^\alpha_\mu + \sqrt{-g}g^{\mu\nu}(\partial_\mu\phi)\delta^\alpha_\nu\right) \tag{3}$$

$\mu \leftrightarrow \nu$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}(\partial_\nu\phi)) + m^2\phi = 0 \tag{4}$$

$$(\Box + m^2)\phi = 0 \tag{5}$$

- A linear partial differential equation
- Set of classical solutions form a linear space

Conjugate Momentum

$$\Pi(x) = \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial\dot{\phi}}$$

The definition of conjugate momentum requires a choice of time coordinate but as we are dealing with a general spacetime there's no canonical choice of time (because coordinates can get mixed up). But for now we take some choice of time

$$t = x^0$$

We'll later find the relation with a coordinate system where choice of time is different
Using above definition

$$\Pi(x) = \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\partial_0\phi)} = \sqrt{-g}g^{0\mu}\partial_\mu\phi(x) \tag{6}$$

$$\tag{7}$$

1.2 Annhiliation and Creation Operators

Commutation Relations

$$[\phi(t,x), \Pi(t,x')] = i\delta(x - x') \tag{8}$$

The solution of 5 can be expressed as

$$\phi = \Sigma_i[(a_i)f_i(t,x) + (a_i^\dagger)f_i^*(t,x)] \tag{9}$$

- For now a_i and $a_i^\dagger$ are just two arbitary constants

Using 8 and 9 we get the following relations

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{10}$$

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{11}$$

$$\Sigma_i\left[f_i(t,x)g^{0\mu}\partial_\mu f_i^*(t,x') - f_i^*(t,x)g^{0\mu}\partial_\mu f_i(t,x')\right] = \frac{i\delta(x - x')}{\sqrt{-g}} \tag{12}$$

Changing the Basis

If Ω_a is a vaccume state then

$$a_i |\Omega_a\rangle = 0 \tag{13}$$

Consider another basis

$$\phi = \Sigma_i[(b_i)g_i(t,x) + (b_i^\dagger)g_i^*(t,x)] \tag{14}$$
$$a_i = \Sigma_j(\alpha_{ji}b_j + \beta_{ji}^* b_j^\dagger) \tag{15}$$
$$a_i^\dagger = \Sigma_j(\alpha_{ji}^* b_j^\dagger + \beta_{ji}b_j) \tag{16}$$
$$\alpha_{ji}, \beta_{ji}^* \longrightarrow \text{Bogoliubov Coefficients}$$

Tutorial Problems

(i) In terms of α and β find relation b/w f_i and g_i
Answer:

$$g_j = \Sigma_j(\alpha_{ij}f_i + \beta_{ij}f_i^*) \tag{17}$$
$$g_j^* = \Sigma_j(\alpha_{ij}^* f_i^* + \beta_{ij}^* f_i) \tag{18}$$

(ii) If $[b_i, b_j^\dagger] = \delta_{ij}$ find constrain on α and β

Just like Ω_a we can define Ω_b such that

$$b|\Omega_b\rangle = 0.$$

$$b_i^\dagger b_k^\dagger |\Omega_b\rangle \longrightarrow \text{fock space}$$

The two vaccume states Ω_a and Ω_b might not be the same. We try to establish a relation between them

$$a_i |\Omega_a\rangle = 0 \tag{19}$$
$$\Sigma_j(\alpha_{ji}b_j + \beta_{ji}^* b_j^\dagger)|\Omega_a\rangle = 0 \tag{20}$$

Ansatz :

$$|\Omega_a\rangle = e^{\frac{1}{2}b_j^\dagger C_{jk} b_k^\dagger}|\Omega_b\rangle$$

$$\Sigma_j(\alpha_{ji}b_j)\left|\Omega_a\right\rangle = \Sigma_{j,m}\alpha_{ji}C_{mj}b_m^\dagger\left|\Omega_a\right\rangle \tag{21}$$
$$\implies (\Sigma_{j,m}\alpha_{ji}C_{mj}b_m^\dagger + \beta_{ji}^* b_j^\dagger)\left|\Omega_a\right\rangle = 0 \tag{22}$$
$$\implies \Sigma_{j,m}\alpha_{ji}C_{mj}b_m^\dagger = -\Sigma_m\beta_{mi}^* b_m^\dagger \tag{23}$$
$$\implies \Sigma_j C_{mj}\alpha ji = -\beta_{mi}^* \tag{24}$$
$$\boxed{C = -\beta^*\alpha^{-1}} \tag{25}$$

From the ansatz we can see that the number of particles are oobserver dependent because the vacuum state $\left|\Omega_a\right\rangle$ is actually an excitation of $\left|\Omega_b\right\rangle$ vaccume state.

Question: If number of particles are observer dependent, what is invariant then? **Answer:** Correlation Functions

$$\langle\phi(x_1)\phi(x_2)\rangle$$

Note: $||\phi(x)||^2$ may depend on frame because renormalization may depend on frame

Now we go back to flat 2d space and use these generalities

1.3 Rindler Coordinates

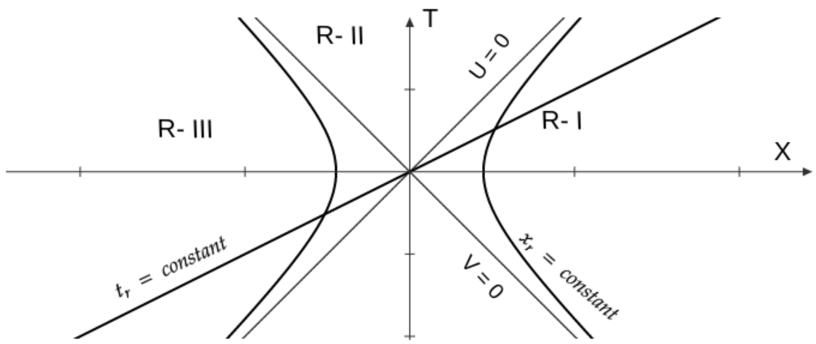

Figure 1: Rindler Coordinates

Consider the 2d Minkowski metric

$$ds^2 = -dT^2 + dX^2$$

We perform a series of coordinate transformations as follows

$$
\begin{aligned}
&U = T - X = -e^{-aU_r} \\
&V = T + X = e^{aV_r} \\
&ds^2 = e^{a(V_r - U_r)} dU_r dV_r \\
&U_r = t_r - x_r \\
&V_r = t_r + x_r \\
&t_r, x_r \longrightarrow \text{Rindler Coordinates} \\
&ds^2 = (e^{2ax_r})(dt_r^2 - dx_r^2)
\end{aligned} \tag{26}
$$

Note that lines of constant x_r represent uniformly accelerated observers moving with different accelerations. (The acceleration is defined as the change in velocity in Momentarily Co-moving Rest Frame(MCRF) i.e. $\frac{dv'}{d\tau}$)
Unruh Effect: Accelerating observer sees a thermal bath of particles
(Details discussed in next lecture)

Wave Equation in Rindler Coordinates

To simplify the analysis we set m=0

$$
\begin{aligned}
&\partial_\mu(\sqrt{-g} g^{\mu\nu}(\partial_\nu \phi)) = 0 \\
&g^{x_r x_r} = -g^{t_r t_r} = e^{-2ax_r} \\
&\sqrt{-g} = e^{2ax_r} \\
&\left(\frac{\partial^2}{\partial x_r^2} - \frac{\partial^2}{\partial t_r^2}\right)\phi = 0 \qquad \text{(Plane Wave Equation)}
\end{aligned} \tag{27}
$$

$$
\phi = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[a_\omega e^{-i\omega U_r} + b_\omega e^{-i\omega V_r} + \text{harmonic conjugates}\right] \tag{28}
$$

- Terms with $U_r = t_r - x_r (V_r = t_r + x_r)$ represent a right(left) moving observer
- The above solutions are just plane wave solutions with $k = \omega$ because m=0

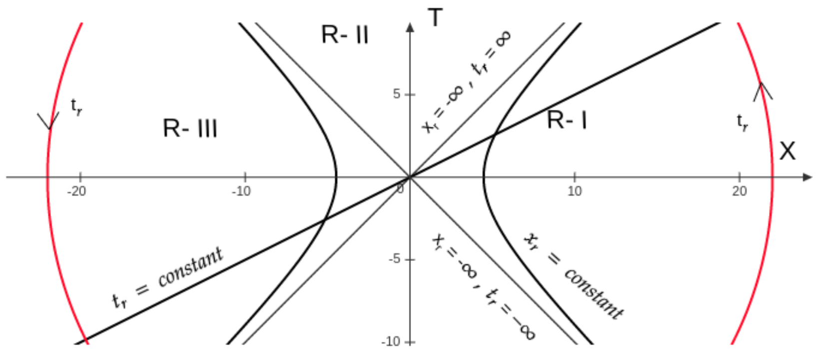

Figure 2: Rindler Coordinates in III

Since the time coordinate t_r is reversed in region III, we can extend our solution 28 to III by flipping the sign of ω

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[\{ \tilde{a}_\omega e^{i\omega U_r} + \tilde{b}_\omega e^{i\omega V_r} + \text{harmonic conjugates} \right] \tag{29}$$

We define the following functions

$$\begin{aligned} U_L(U_r) &= e^{i\omega U_r} && \text{(R-III)} \\ &= 0 && \text{(R-I)} \end{aligned}$$

$$\begin{aligned} U_R(U_r) &= 0 && \text{(R-III)} \\ &= e^{-i\omega U_r} && \text{(R-I)} \end{aligned}$$

$$\begin{aligned} V_L(V_r) &= e^{i\omega V_r} && \text{(R-III)} \\ &= 0 && \text{(R-I)} \end{aligned}$$

$$\begin{aligned} V_R(V_r) &= 0 && \text{(R-III)} \\ &= e^{-i\omega V_r} && \text{(R-I)} \end{aligned}$$

A solution valid for both I and III

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[\tilde{a}_\omega U_L(U_r) + \tilde{b}_\omega V_L(V_r) + a_\omega U_R(U_r) + b_\omega V_R(V_r) + \text{harmonic conjugates} \right] \tag{30}$$

Since the wave equation has the same form as 27 we can write solutions of the same form in Minkowski Coordinates

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[C_\omega e^{-i\omega(T-X)} + D_\omega e^{-i\omega(T+X)} + \text{harmonic conjugates} \right] \quad (31)$$

Now we'll try to evaluate Bogoliubov coefficients between the Rindler and Minkowski Coordinates

2 Lecture 6: The Unruh Effect

2.1 Unruh Modes

We have these two expansions

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[\tilde{a}_\omega U_R(U_r) + \tilde{b}_\omega V_R(V_r) + a_\omega U_L(U_r) + b_\omega V_L(V_r) + \text{harmonic conjugates}\right] \tag{32}$$

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[C_\omega e^{-i\omega(T-X)} + D_\omega e^{-i\omega(T+X)} + \text{harmonic conjugates}\right] \tag{33}$$

Finding full Bogoliubov coefficients is a bit difficult so we'll use a trick employed by Unruh.
We need to find $|\Omega_M\rangle$ in terms of $|\Omega_{Rind}\rangle$
We consider a third expansion in terms of **Unruh Modes**

$$\begin{aligned} U_U(U_r) &= U_L^*(U_r) && \text{(R-III)} \\ &= e^{\frac{\pi\omega}{a}} U_R(U_r) && \text{(R-I)} \end{aligned}$$

$$\begin{aligned} U^{\frac{i\omega}{a}} &= e^{-i\omega U_r} = U_L^*(U_r) && \text{(R-III)} \\ U^{\frac{i\omega}{a}} &= (-e^{-aU_r})^{\frac{i\omega}{a}} = (e^{-i\pi} e^{-aU_r})^{\frac{i\omega}{a}} = e^{\frac{\pi\omega}{a}} U_R(U_r) && \text{(R-I)} \end{aligned}$$

Note: $U \propto e^{-U_r}$ in $R - III$
If we choose the branch cut to be in upper half plane (U_u function is analytic in lower half plane) of of U-V plane then $-1 = e^{-i\pi}$ and with this choice

$$U_U(U_r) = U^{\frac{i\omega}{a}} \qquad \text{(R-I and R-III)} \tag{34}$$

We could've also chosen the following mode with choice of branch cut in upper half plane

$$\begin{aligned} \tilde{U}_U(U_r) &= U_L(U_r) && \text{(R-III)} \\ &= e^{\frac{-\pi\omega}{a}} U_R^*(U_r) && \text{(R-I)} \end{aligned}$$

The modes $U_u(U_r)$ have the following property

$$\int_{-\infty}^{\infty} U_u(T, X) e^{-i\omega' T} dT = 0 \qquad \forall \omega' > 0 \tag{35}$$

This is becuase both U_u and $e^{-i\omega' T}$ are analytic in lower half plane (need to be precise) Also notice that above integral is a inverse fourier transform of

$U_U(X,T)$ for a negative frequency $-\omega'$. This integral equal to 0 implies that there are no negative frequency modes present in $U_U(X,T)$. This can be expressed as

$$U(T,X) = \int_0^\infty \chi(\omega) e^{-i\omega T} d\omega \tag{36}$$

$$\tilde{U}(T,X) = \int_0^\infty \tilde{\chi}(\omega) e^{-i\omega T} d\omega \tag{37}$$

where the integral from 0 to ∞ implies the presence of only positive frequency Minkowski modes

$$\phi = \int \frac{d\omega}{\sqrt{\omega}} \left[e_\omega U_U(U_r) + \tilde{e}_\omega \tilde{U}_U(U_r) + f_\omega V_U(V_r) + \tilde{f}_\omega \tilde{V}_U(V_r) + hc \right] \tag{38}$$

This is the Unruh Expansion

Tutorial Problems

(i) Define $V_U(V_r)$ and $\tilde{V}_U(V_r)$
Answer:

$$V_U(V_r) = e^{\frac{\pi\omega}{a}} V_L^*(U_r) \quad \text{(R-III)}$$
$$= V_R(V_r) \quad \text{(R-I)}$$
$$\tilde{V}_U(V_r) = e^{\frac{-\pi\omega}{a}} V_L(U_r) \quad \text{(R-III)}$$
$$= V_R^*(V_r) \quad \text{(R-I)}$$

Comparing 36 and 17 we get that $\beta^* = 0$ as the postive frequency Unruh Modes are composed only of positive frequency Minkowski Modes. From 15 we also observe that Unruh annhiliation operators conist only of Minkowski annhiliation operatos (and not Minkowski creation operators). Thus we can conclude that the Vaccum states for both Unruh and Minkowski modes are the same The Minkowski Vaccum satisfies

$$C_\omega |\Omega_m\rangle = D_\omega |\Omega_m\rangle = 0$$

also satisfies

$$e_\omega |\Omega_m\rangle = \tilde{e}_\omega |\Omega_m\rangle = f_\omega |\Omega_m\rangle = \tilde{f}_\omega |\Omega_m\rangle = 0$$

$$\boxed{\text{Minkowski Vaccum} = \text{Unruh Vaccum}} \tag{39}$$

Comparing 38 and 32 we can write the following relations

$$a_\omega = e^{\frac{\pi\omega}{a}} e_\omega + ^{\frac{\pi\omega}{a}} \tilde{e}^\dagger_\omega \tag{40}$$

$$\tilde{a}_\omega = e^\dagger_\omega + \tilde{e}_\omega \tag{41}$$

$$\tilde{a}^\dagger_\omega = e_\omega + \tilde{e}^\dagger_\omega \tag{42}$$

$$\implies e_\omega = \frac{a_\omega - e^{-\pi\omega/a}\tilde{a}^\dagger_\omega}{e^{\pi\omega/a} - e^{-\pi\omega/a}} \tag{43}$$

$$\implies \tilde{e}_\omega = \frac{\tilde{a}_\omega - e^{-\pi\omega/a}a^\dagger_\omega}{1 - e^{-2\pi\omega/a}} \tag{44}$$

Similarly

$$f_\omega = \frac{b^\dagger_\omega - e^{-\pi\omega/a}\tilde{b}_\omega}{e^{\pi\omega/a} - e^{-\pi\omega/a}} \tag{45}$$

$$\tilde{f}_\omega = \frac{\tilde{b}^\dagger{}_\omega - e^{-\pi\omega/a}b_\omega}{1 - e^{-2\pi\omega/a}} \tag{46}$$

Assymetries in the denominator as we have not normalised $e_\omega, f_\omega, \tilde{e}_\omega,$ and $\tilde{f}_\omega$

2.2 Relating the Rindler and Minkowski Vaccums

Using the ansatz $|\Omega_a\rangle = e^{\frac{1}{2}b^\dagger_j C_{jk} b^\dagger_k}|\Omega_b\rangle$ we can calculate the relation between the two vaccum states $|\Omega_U\rangle$ and $|\Omega_{Rind}\rangle$

First we calculate the matrices β and α for a particular frequency ω operators. (κ is the normalisation constant)

$$\alpha_\omega = \begin{array}{c} \\ e_\omega \\ \tilde{e}_\omega \\ f_\omega \\ \tilde{f}_\omega \end{array} \begin{array}{c} \begin{matrix} a_\omega & \tilde{a}_\omega & b_\omega & \tilde{b}_\omega \end{matrix} \\ \begin{pmatrix} \frac{1}{\kappa} & 0 & 0 & 0 \\ 0 & \frac{1}{\kappa} & 0 & 0 \\ 0 & 0 & 0 & \frac{-e^{-\pi\omega/a}}{\kappa} \\ 0 & 0 & \frac{-e^{-\pi\omega/a}}{\kappa} & 0 \end{pmatrix} \end{array} \tag{47}$$

$$\beta_\omega = \begin{array}{c} \\ e_\omega \\ \tilde{e}_\omega \\ f_\omega \\ \tilde{f}_\omega \end{array} \begin{array}{c} \begin{array}{cccc} a_\omega^\dagger & \tilde{a}_\omega^\dagger & b_\omega^\dagger & \tilde{b}_\omega^\dagger \end{array} \\ \begin{pmatrix} 0 & \frac{-e^{-\pi\omega/a}}{\kappa} & 0 & 0 \\ \frac{-e^{-\pi\omega/a}}{\kappa} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\kappa} & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa} \end{pmatrix} \end{array} \tag{48}$$

$$C_\omega = -\,\beta_\omega \alpha_\omega^{-1} \tag{49}$$

$$\begin{array}{c} \\ a_\omega^\dagger \\ \tilde{a}_\omega^\dagger \\ b_\omega^\dagger \\ \tilde{b}_\omega^\dagger \end{array} \begin{array}{c} \begin{array}{cccc} a_\omega^\dagger & \tilde{a}_\omega^\dagger & b_\omega^\dagger & \tilde{b}_\omega^\dagger \end{array} \\ \begin{pmatrix} 0 & e^{-\pi\omega/a} & 0 & 0 \\ e^{-\pi\omega/a} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\pi\omega/a} \\ 0 & 0 & e^{-\pi\omega/a} & 0 \end{pmatrix} \end{array} \tag{50}$$

$$\implies \boxed{|\Omega_M\rangle = e^{\int d\omega e^{-\pi\omega/a}(a_\omega^\dagger \tilde{a}_\omega^\dagger + b_\omega^\dagger \tilde{b}_\omega^\dagger)}\,|\Omega_{\mathsf{I,III}}\rangle} \tag{51}$$

The above exponential operator when expanded gives superposition of Unruh states with different energies with a factor of $e^{-\pi\omega/a} = e^{-(2\pi/a)}E$. As the raising operators $a_\omega^\dagger$ and $\tilde{a}_\omega^\dagger$act in pair, for any operator operation the energy of the state becomes 2ω. Finally we obtain

$$|\Omega_M\rangle = \frac{1}{\sqrt{Z}}\Sigma_E e^{-\beta E/2}\,|E_I, E_{III}\rangle \tag{52}$$

where

$$E_I = \int d\omega\omega(a_\omega a_\omega^\dagger + b_\omega b_\omega^\dagger) \qquad \text{(Total energy in I)} \tag{53}$$

$$E_{III} = \int d\omega\omega(\tilde{a}_\omega \tilde{a}_\omega^\dagger + \tilde{b}_\omega \tilde{b}_\omega^\dagger) \qquad \text{(Total energy in III)} \tag{54}$$

$$\beta = \frac{2\pi}{a} \tag{55}$$

We can use the above expression to define the density matrix for I observer

$$\rho_I = \frac{1}{Z}e^{-\beta H} = \frac{1}{Z}\Sigma_E e^{-\beta E}\,|E_I\rangle\langle E_I| \tag{56}$$

To complete the picture we compute the expansion in region-II as well.
In region- II

$$U = e^{-aU_r} \quad V = e^{aV_r}$$

$$dUdV = \frac{1}{a^2} e^{a(V_r - U_r)}(-dU_r dV_r) \tag{57}$$

$$= \frac{1}{a^2} e^{2ax_r}(dx_r^2 - dt_r^2) \tag{58}$$

Here x_r is the timelike coordinate. In the expansion of ϕ we want the terms of the form $e^{-i\omega x_r}$ (because x_r increases in the direction of T[just like t_r in I] and this is why in IV we use $e^{i\omega x_r}$[just like t_r in III]) thus we get the following expansion

$$\phi_{II} = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[\{\tilde{a}_\omega e^{i\omega U_r} + b_\omega e^{-i\omega V_r} + \text{harmonic conjugates}\right] \tag{59}$$

Key Takeaway: In region II by continuity the "V" modes (left movers) from the right and "U" modes (right movers) from the left cross over the horizon

2.3 Hawking Radiation

- Analogy of Rindler Horizon with Schwarzschild Horizon
- Schwarzschild observer (an observer at rest at a fixed r) is also accelerating from Equivalence Principle

We write the wave equation in Schwarzschild coordinates

$$\begin{aligned} ds^2 &= (1-\frac{2m}{r})dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2 d\Omega^2 &(60)\\ &= (1-\frac{2m}{r})(dt^2 - dr_*^2) - r^2 d\Omega^2 &(61)\end{aligned}$$

$$\begin{aligned} &\sqrt{-g} = (r-2m)\\ &g^{tt} = g^{r_* r_*} = \frac{1}{1-\frac{2m}{r}}\\ &\Box\phi = 0\\ &\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}(\partial_\nu\phi)) = 0\\ \implies &\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2}\right)\phi = 0 \qquad \text{(Near Horizon)} \qquad (62)\end{aligned}$$

Note: The above equations holds irrespective of mass of the field, interactions (i.e. a $\Box\phi = V(\phi)$) and angular momentum. This is because the $\sqrt{-g}$ term goes to zero near horizon and therefore it gets multiplied to the mass and interaction term on RHS of wave equation.
The solutions to the above wave equation are

$$\begin{aligned}\phi &= e^{-i\omega(t-r_*)}\\ &= e^{-i\omega(t+r_*)}\end{aligned}$$

$$\phi = \int \frac{d\omega}{\sqrt{\omega}}\left[a_\omega e^{-i\omega(t-r_*} + b_\omega e^{-i\omega(t+r_*)} + hc\right] Y_m(\Omega) \qquad \text{(Near Horizon Solution)} \qquad (63)$$

We assume that an infalling observer sees the vaccum as is predicted from the equivalence principle. The observers see the Kruskal UV coordinates near the horizon that are locally flat. Locally the relation between U, V, r_*, t is given by

$$U_K = -e^{(r_*-t)/4M}, V_K = e^{(r_*+t)/4M}$$

$$ds^2 = \frac{32m^3}{r} dU_K dV_k + r^2 d\Omega^2 \tag{64}$$
$$T_k = (U_k + V_k)/2 \qquad X_k = V_k - U_k)/2 \tag{65}$$
$$ds^2 = \frac{32m^3}{r}(dT_k^2 - dX_k^2) + r^2 d\Omega^2$$
$$\phi_k = \int \frac{d\omega}{\sqrt{\omega}} \left[c_\omega e^{-i\omega U_K} + b_\omega e^{-i\omega V_K} + hc\right] Y_m(\Omega) \qquad \text{(Near Horizon Solution)} \tag{66}$$

- Physics looks exactly like 2d
- Transformation from Kruskal to Schwarschild coordinates is exactly like Minkowski to Rindler

Using Rindler

$$|\Omega_{\text{infalling}}\rangle = \text{thermal bath for a-modes}$$

$$|\Omega_{\text{inf}}\rangle = e^{\int d\omega e^{-\pi\omega/4M}(a_\omega^\dagger \tilde{a}_\omega^\dagger)} |\Omega_a\rangle \tag{67}$$
$$\implies \boxed{\beta = 8\pi M} \tag{68}$$

3 Lecture 7: Hawking Radiation

3.1 Hawking's Original Derivation

- A geometrical derivation
- Start off with a vaccum state in the far past and observers in the far future observe a thermal bath of particles

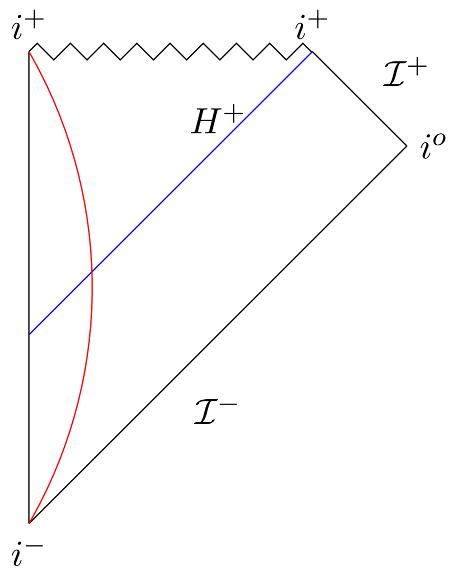

Figure 3: Oppenheimer-Snyder (Collapsing) Black Hole

Initial data can be specified

1. data on $\mathcal{I}^-$
2. data on $\mathcal{I}^+ \cup H^+$

Note: We consider null infinites because we are considering massless particles
So we have two possible expansions of the fields

$$\phi = \Sigma_i a_i f_i(r, t, \Omega) + hc \tag{69}$$

$$\Sigma_i a_i f_i \xrightarrow{\mathcal{I}^-} \Sigma_m \int \frac{d\omega}{\sqrt{\omega}} a_\omega \frac{e^{-i\omega V}}{r} Y_m(\Omega) + hc \tag{70}$$

- 1/r because of spherical waves

- $e^{-i\omega V}$ because waves are incoming from past null infinity

$$\phi = \Sigma_i b_i g_i(r,t,\Omega) + c_i h_i(r,t,\Omega) + hc \tag{71}$$

$$\Sigma_i b_i g_i \xrightarrow{\mathcal{I}^+} \Sigma_m \int \frac{d\omega}{\sqrt{\omega}} b_\omega \frac{e^{-i\omega U}}{r} Y_m(\Omega) + hc \tag{72}$$

- h_i solution is corresponding to the future rays ending up at horizon and are not of much interest

Note: Here we have considered two different solutions also because of a time dependent geometry. Due to the collapse the geometry becomes time dependent.

Question: Find Bogoliubov transforms between a_ω, b_ω and c_ω
Hawking's Insight: $a_\omega \to b_\omega$ Bogoliubov coefficients do not depend on the details of the collapse.

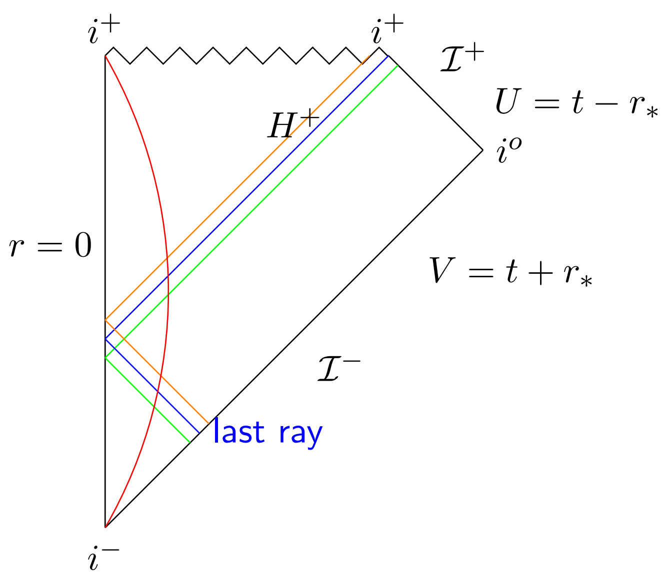

Figure 4: Geometrical Optics Approach by Stephen Hawking

We consider light rays moving towards the origin r=0 from the past null infinity $\mathcal{I}^-$ getting reflected from the origin and then moving towards future null infiniy $\mathcal{I}^+$. At each point on the penrose diagram we have a suppressed 2d sphere so actually instead of light rays we have collapsing shells of light which contract uptil r=0 and then again start expanding. We can consider three types of null rays

(i) Blue light ray - These are the rays which end up getting stuck at the event horizon of the black hole.

(ii) Green light rays - These are the rays which escae the event horizon and end up at future null infinity.

(iii) Orange light rays - These are the rays which get trapped inside the horizon and end up at the spacelike singularity

$$|\delta x| = \delta\tau \qquad \text{Inside Shell}$$
$$|\delta r_*| = \delta t \qquad \text{Outisde Shell}$$

Reason: Because inside the shell the effect of shell's gravity is negligibe and hence the spacetime looks like Minkowski. While outside the shell spacetime looks like a Schwarschild one.

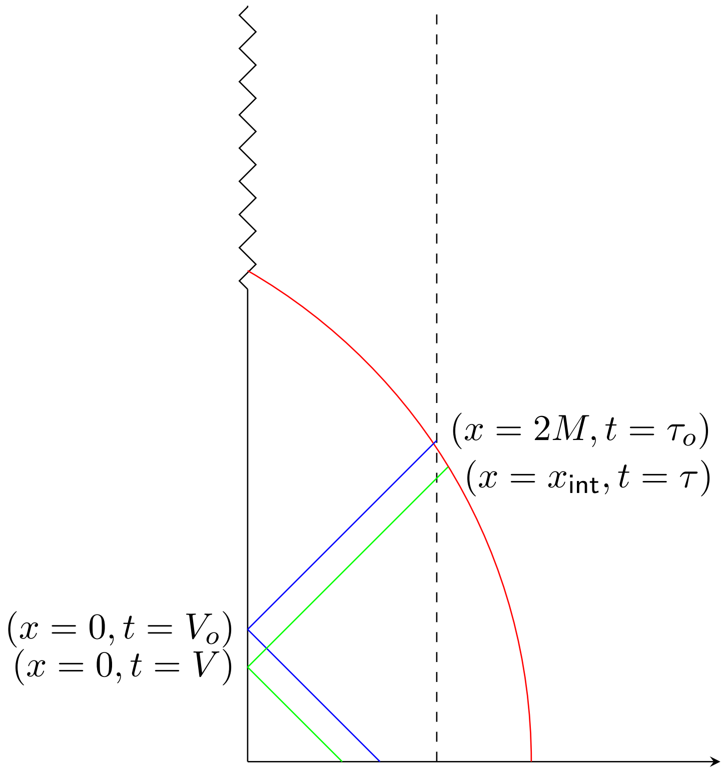

Figure 5: Geometrical Optics Approach by Stephen Hawking

V_s = V-coordinate of surface of shell seen from outside
α = inverse velocity of shell

$$
\begin{aligned}
&\alpha(x_{\text{int}} - 2M) = \tau_o - \tau \\
&\tau = \tau_o - \alpha(x_{\text{int}} - 2M) \\
&\tau = V_o + 2M - \alpha(x_{\text{int}} - 2M) \\
&V = \tau - x_{\text{int}} = V_o - (1+\alpha)(x_{\text{int}} - 2M) \\
&r_* = (V_s - U_{\text{ray}})/2 \\
\implies &U_{\text{ray}} = V_s - 2r_*
\end{aligned}
\tag{73}
$$

$$r_* = 2M + 2M \ln \left| \frac{x_{\text{int}} - 2M}{2M} \right| \quad (74)$$

$$r_* \approx 2M + 2M \ln \frac{V_o - V}{2M(1+\alpha)} \quad (75)$$

$$\boxed{U_{\text{ray}} = V_s - 4M + 4M \ln \frac{V_o - V}{2M(1+\alpha)}} \quad (76)$$

- The result 76 is the central ray tracing result!
- Note that the constants $V_s - 4m$ and $2m(1+\alpha)$ are irrelevant constants and can be absorbed in the definition of origin of U coordinates
- From 76 we notice that a small ΔV at $V = V_o$ can produce a large change in U coordinate

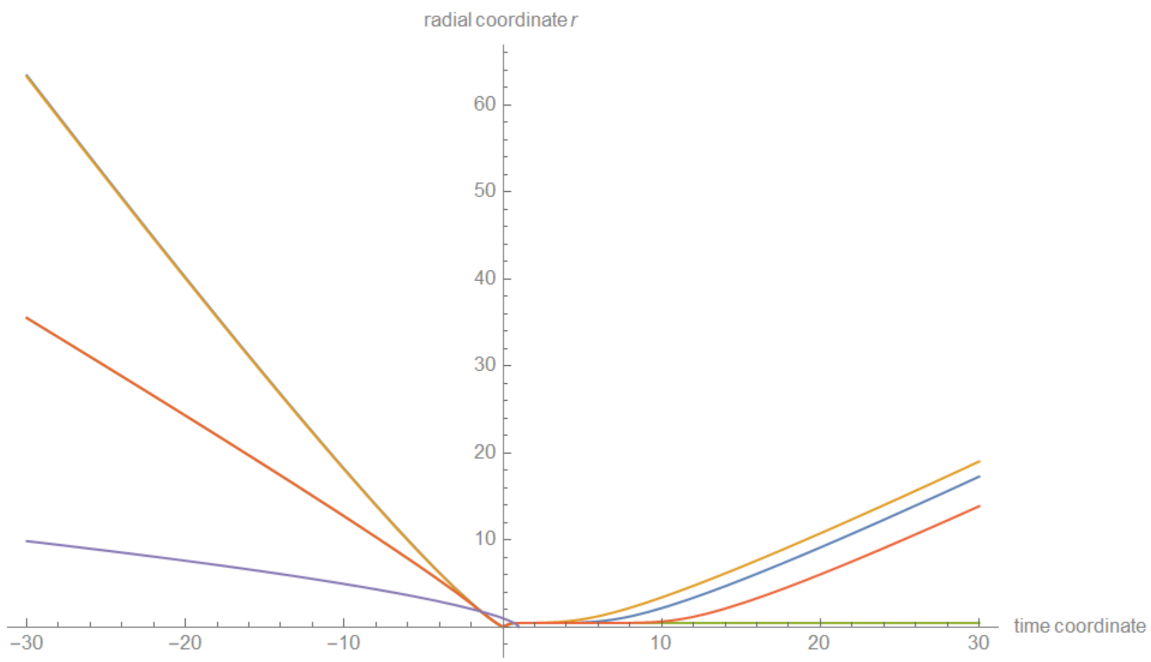

Figure 6: Purple curve denotes the surface of the star. Notice the incoming rays with almost same V coordinate travel along widely different U coordinates after reflection (Source: http://www.suvratraju.net/classes/serc-school-black-holes-and-information/materials/lecture-7)

$$\phi_{\text{in}}(V) = \Sigma_m \int \frac{d\omega}{\sqrt{\omega}} a_\omega \frac{e^{-i\omega V}}{r} Y_m(\Omega) + hc \tag{77}$$

$$\phi_{\text{out}}(U) = \Sigma_m \int \frac{d\omega}{\sqrt{\omega}} b_\omega \frac{e^{-i\omega U}}{r} Y_m(\Omega) + hc \tag{78}$$

and

$$U \sim 4M \ln(|V_o - V|)$$

The trasnformation b/w U and V coordinates look exactly like those between Rindler and Minkowski Coordinates. Thus the vaccum for a_ω modes (Minkowski like vaccum) looks like a thermal bath of photons for b_ω modes (Rindler like modes) with density matrix

$$\rho_{\mathcal{I}^+} \sim \frac{1}{Z} e^{-\beta H}$$

where $\beta = 8\pi M$

3.2 Derivation using correlators

$$\langle \phi(U_1, V_1, x_1)\phi(U_2, V_2, x_2)\rangle$$

$$\lim_{|y_1 - y_2|^2 \to 0} \langle \phi(y_1)\phi(y_2)\rangle \propto \frac{1}{g^{\mu\nu}(y_{1_\mu} - y_{2_\mu})(y_{1_\nu} - y_{2_\nu})} \tag{79}$$

The result above is a very general result for points y_1 and y_2 approaching a light cone. Now the metric in U,V,x coordinates look like

$$ds^2 = kdUdV - (dx)^2 \tag{80}$$

We consider two points one outside (U_1, V) and the other inside $(U_2, V + \delta V)$ the horizon. As $\delta V \to 0$ the two points approach a light cone. Now we have

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{k\delta U \delta V - (\delta x)^2} \tag{81}$$

$$\langle \partial_{U_1}\phi_1 \partial_{U_2}\phi_2 \rangle = \frac{-2k^2(\delta V)^2}{(k\delta U\delta V - (\delta x)^2)^3} \tag{82}$$

$$\lim_{\delta V \to 0} \langle \partial_{U_1}\phi_1 \partial_{U_2}\phi_2 \rangle \propto \frac{\delta^2(\delta x)}{(\delta U)^2} \tag{83}$$

$$\langle \partial_{U_1}\phi(U_1, V_1, x_1) \partial_{U_2}\phi(U_2, V_2, x_2)\rangle \propto \frac{\delta^2(x_1 - x_2)}{(U_1 - U_2)^2} \tag{84}$$

The transverse delta function shows ultralocality in the transverse directions

We can write the solutions ϕ_{outside} and ϕ_{inside}(Recall 59) as follows

$$\phi_{\text{outside}} = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[a_\omega V^{-i\beta\omega/2\pi} + b_\omega(-U_2)^{i\beta\omega/2\pi} + \text{harmonic conjugates}\right] e^{ik.x_1} \tag{85}$$

$$\phi_{\text{inside}} = \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[a_\omega V^{-i\beta\omega/2\pi} + \tilde{b}_\omega U_1^{-i\beta\omega/2\pi} + \text{harmonic conjugates}\right] e^{ik.x_2} \tag{86}$$

Note: The above expansions are different from what's given in Prof Suvrat's notes as in his notes $U_1^{i\beta\omega/2\pi}((-U_2)^{-i\beta\omega/2\pi})$ instead of $U_1^{-i\beta\omega/2\pi}((-U_2)^{i\beta\omega/2\pi})$ is written which is probably a sign mistake

We can prove if

$$\langle b_\omega \tilde{b}_{\omega'}\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\delta(\omega-\omega')$$

holds then 84 is satisfied. The proof is as follows

$$\langle \partial_{U_1}\phi_1 \partial_{U_2}\phi_2\rangle = \int d\omega d\omega' dk dk' \sqrt{\omega\omega'}\frac{4\pi^2\beta^2}{U_1 U_2}\langle[(\tilde{b}_\omega U_1^{-i\beta\omega/2\pi} + \tilde{b}^\dagger_\omega U_1^{i\beta\omega/2\pi})e^{ikx_1}] \tag{87}$$
$$[(b_{\omega'}(-U_2)^{i\beta\omega'/2\pi} + b^\dagger{}_{\omega'}(-U_2)^{-i\beta\omega'/2\pi})e^{ik'x_2}]\rangle$$

Substitute

$$\left\langle b_{\omega k}\tilde{b}_{\omega' k'}\right\rangle = \left\langle b^\dagger_{\omega k}\tilde{b}^\dagger_{\omega' k'}\right\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta(k+k') \tag{88}$$

$$\left\langle b_{\omega k}\tilde{b}^\dagger_{\omega' k'}\right\rangle = \left\langle b^\dagger_{\omega k}\tilde{b}_{\omega' k'}\right\rangle = 0 \tag{89}$$

$$
\begin{aligned}
\langle \partial_{U_1}\phi_1 \partial_{U_2}\phi_2\rangle &= \int d\omega d\omega' \sqrt{\omega'\omega}\frac{4\pi^2\beta^2\delta^2(x_1-x_2)}{U_1U_2}\cdot \\
&\quad \langle (\tilde{b}_\omega U_1^{-i\beta\omega/2\pi} - \tilde{b}^\dagger_\omega U_1^{i\beta\omega/2\pi})(b_{\omega'}(-U_2)^{i\beta\omega'/2\pi} - b^\dagger{}_{\omega'}(-U_2)^{-i\beta\omega'/2\pi})\rangle \\
&= \int d\omega d\omega' \sqrt{\omega'\omega}\frac{4\pi^2\beta^2\delta^2(x_1-x_2)}{U_1U_2}\cdot \\
&\quad \left(\langle \tilde{b}_\omega b_{\omega'}\rangle (U_1^{-i\beta\omega/2\pi}\cdot(-U_2)^{i\beta\omega'/2\pi}) + \langle \tilde{b}^\dagger_\omega b^\dagger_{\omega'}\rangle (U_1^{i\beta\omega/2\pi}\cdot(-U_2)^{-i\beta\omega'/2\pi})\right) \\
&= \int_{-\infty}^{\infty} d\omega\omega \frac{4\pi^2\beta^2\delta^2(x_1-x_2)}{U_1U_2}\cdot\frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\left(\left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi} + \left(\frac{-U_2}{U_1}\right)^{i\beta\omega/2\pi}\right) \\
&= 2\int_{-\infty}^{\infty} d\omega\omega \frac{4\pi^2\beta^2\delta^2(x_1-x_2)}{U_1U_2}\cdot\frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi}
\end{aligned}
$$

If $|U_1| > |U_2|$, we complete the contour through the upper half plane, otherwise through the lower half plane.
Picking up the poles at $\beta\omega = 2in\pi$ we get

$$
\begin{aligned}
\int d\omega\omega \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\left(\frac{U_1}{-U_2}\right)^{i\beta\omega/2\pi} &= -\frac{1}{\beta}\Sigma_n n(-1)^n\left(\frac{U_1}{-U_2}\right)^n \\
&= -\frac{1}{\beta}\frac{U_1U_2}{(U_1-U_2)^2} \qquad \left(\text{Using } \Sigma_n nx^n = \frac{x}{(1-x)^2}\right)
\end{aligned}
$$

Inserting the remaining factors we find that

$$
\langle \partial_{U_1}\phi(U_1,V_1,x_1)\partial_{U_2}\phi(U_2,V_2,x_2)\rangle \propto \frac{\delta^2(x_1-x_2)}{(U_1-U_2)^2}
$$

Hence proving our assumption about correlation function $\left\langle b_{\omega k}\tilde{b}_{\omega' k'}\right\rangle$ correct.
Taking both the points outside the horizon, we can find the following coorelators

$$
\langle b_\omega b^\dagger_{\omega'}\rangle = \frac{1}{1-e^{-\beta\omega}}\delta(\omega-\omega') \tag{90}
$$

$$
\langle b^\dagger_\omega b_{\omega'}\rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\delta(\omega-\omega') \tag{91}
$$

Here 91 can be derived using the commutation relation $[b_\omega, b^\dagger_{\omega'}] = \delta(\omega-\omega')$

Note: At a very fundamental level what we have done is we started with a position space correlator $\langle \partial_{U_1}\phi_1 \partial_{U_2}\phi_2 \rangle$ and fourier transformed it to momentum space correlators $\left\langle b_{\omega k}\tilde{b}_{\omega' k'} \right\rangle$

Conclusion: Two Point Correlators for outgoing b-modes are thermal or thermally populated

4 How close are pure and mixed states?

- Pure State
$$|\Psi\rangle = a_1 |1\rangle + a_2 |2\rangle$$
- Classical Mixture
Prob(Ψ_1) $= |a_1|^2$
Prob(Ψ_2) $= |a_2|^2$

For any observable A we have

$$\langle A\rangle = |a_1|^2 \langle 1| A |1\rangle + |a_2|^2 \langle 2| A |2\rangle + a_2^* a_1 \langle 2| A |1\rangle + a_1^* a_2 \langle 1| A |2\rangle \quad \text{(Pure State)}$$
$$\langle A\rangle = |a_1|^2 \langle 1| A |1\rangle + |a_2|^2 \langle 2| A |2\rangle \quad \text{(Mixed State)} \quad (92)$$
$$\rho = |a_1|^2 |1\rangle \langle 1| + |a_2|^2 |2\rangle \langle 2| \quad \text{(Mixed State)} \quad (93)$$
$$\rho_{\text{pure}} = |\Psi\rangle \langle\Psi| \quad (94)$$
$$\implies \rho_{\text{pure}}^2 = \rho_{\text{pure}} \quad (95)$$
$$\langle A\rangle = tr(\rho A) \quad (96)$$

Now we consider a physical setting where

$$E_o - \Delta \leq E_i \leq E_o + \Delta$$
$$Psi = \sum_{i=1}^{w} a_i |E_i\rangle \quad (97)$$
$$(98)$$

w = Number of eigenstates in interval $[E_o - \Delta,\ E_o + \Delta]$
w = e^S
A particular density matrix

$$\rho_{\text{micro}} = \frac{1}{w} \sum_{i=1}^{w} |E_i\rangle \langle E_i| \quad (99)$$

is known as Microcanonical Density Matrix
Now we ask a question, how close is

$$\Psi = \sum_{i=1}^{w} a_i |E_i\rangle \quad (100)$$
$$(101)$$

to

$$\rho_{\text{micro}} = \frac{1}{w}\sum_{i=1}^{w} |E_i\rangle\langle E_i| \tag{102}$$

"Typical State" $|\Psi\rangle$ are "extremely close" to ρ

Physical Notion of Closeness

From the pov of physical observations how easy or difficult it is to distinguish the two?
Physical Observations $\iff$ Probabilities of different outcomes
P = projector
The main observations we are interested in are

$$\langle\Psi| P|\Psi\rangle$$

Note that a general operator $\hat{O}$ can be expressed as

$$\hat{O} = \sum_i \lambda_i |i\rangle\langle i|$$
$$\hat{O} = \sum_i \lambda_i \hat{P}_i$$

$\langle\Psi| \hat{P}_i|\Psi\rangle$ gives the probability of obtaining λ_i.
So given some projector P, we want to see how

$\langle\Psi| P|\Psi\rangle$ $\quad$ $|\Psi\rangle \longrightarrow$ Typical State

compares with

$tr(\rho P)$ $\quad$ $\rho \longrightarrow$ microcanonical density matrix

Precise meaning of "Typical"

We can pick any state

$$\sum_i a_i |E_i\rangle$$

subject to

$$\sum_i |a_i|^2 = 1$$

The Hilbert space is a copy of D^{w-1}. We can visualise it as a sphere in D^w dimensions where the states (actually the point $(a_1, a_2 \cdots a_w)$) lie on the surface of the sphere. The question is if one picks up a point on the sphere at random, it gives thethe state $|\Psi\rangle$. What do we expect for $\langle P\rangle$ for such a state?

We introduce a probability distribution on the hilbert space

$$d\mu_\Psi = \frac{1}{V}\delta(\Sigma|a_i|^2 - 1)\Pi d^2 a_i \tag{103}$$

Here $d^2 a_i$ is beacuse a_i is a complex number and V is chosen such that $\int d\mu_\Psi = 1$

We then ask about the expectation value and higher moments of

$$\delta = \langle\Psi| P |\Psi\rangle - tr(\rho P) \tag{104}$$

We first compute $\langle\delta\rangle$. Here

$$\langle\langle\Psi| P |\Psi\rangle\rangle = \int \langle\Psi| P |\Psi\rangle\, d\mu_\Psi \tag{105}$$

$$\langle\Psi| P |\Psi\rangle = \sum_i |a_i|^2 \langle E_i| P |E_i\rangle + \sum_{i\neq j} a_i a_j^* \langle E_j| P |E_i\rangle$$

$$\implies \langle\langle\Psi| P |\Psi\rangle\rangle = \sum_i \left(\int |a_i|^2 d\mu_\Psi\right) \langle E_i| P |E_i\rangle + \sum_{i\neq j} \left(\int a_i a_j^* d\mu_\Psi\right) \langle E_j| P |E_i\rangle \tag{106}$$

We use symmetry arguments to compute these integrals. By symmetry

$$\int |a_i|^2 d\mu_\Psi = \int |a_j|^2 d\mu_\Psi \qquad \forall i, j \tag{107}$$

$$\int \sum_i |a_i|^2 d\mu_\Psi = 1 \tag{108}$$

$$\implies \int |a_i|^2 d\mu_\Psi = \frac{1}{w}$$

$$\sum_{j\neq i} \int a_i a_j^* d\mu_\Psi = 0$$

where the second integral follows from the fact that for a given a_i we are summing over all possible values of a_j and thus corresponding to a $a_j^* a_i$ there is a $-a_j^* a_i$ which cancels it. These simplifications give us

$$\langle\langle\Psi| P |\Psi\rangle\rangle = \frac{1}{w}\sum_i \langle E_i| P |E_i\rangle = \int tr(\rho P) d\mu_\Psi = tr(\rho P)$$
$$\implies \langle\delta\rangle = 0! \tag{109}$$

We can also compute $\langle\delta^2\rangle$

$$\langle\delta^2\rangle = \int \left(\langle\Psi| P |\Psi\rangle - tr(\rho P)\right)^2 d\mu_\Psi$$
$$\langle\delta^2\rangle = \int \left(\langle\Psi| P |\Psi\rangle\right)^2 d\mu_\Psi - tr(\rho P)^2$$
$$\langle\Psi| P |\Psi\rangle = \sum_i\sum_j \langle E_j| P |E_i\rangle \left(a_i a_j^*\right)$$
$$\int \left(\langle\Psi| P |\Psi\rangle\right)^2 d\mu_\Psi = \sum_k\sum_l\sum_i\sum_j \int a_i a_j^* a_k a_l^* d\mu_\Psi \langle E_j| P |E_i\rangle \langle E_l| P |E_k\rangle$$
$$\int a_i a_j^* a_k a_l^* d\mu_\Psi = (\delta_{ij}\delta kl + \delta_{il}\delta_{kj})\frac{1}{w(w+1)} \longrightarrow \text{Doubt}$$
$$\int \left(\langle\Psi| P |\Psi\rangle\right)^2 d\mu_\Psi = \sum_k\sum_l\sum_i\sum_j (\delta_{ij}\delta kl + \delta_{il}\delta_{kj})\frac{1}{w(w+1)} \langle E_j| P |E_i\rangle \langle E_l| P |E_k\rangle$$
$$\int \left(\langle\Psi| P |\Psi\rangle\right)^2 d\mu_\Psi = \frac{1}{w(w+1)}\sum_k\sum_i \left(\langle E_k| P |E_k\rangle\right)\left(\langle E_i| P |E_i\rangle\right) + \frac{1}{w(w+1)}\sum_i\sum_j \langle E_i| P |E_j\rangle \langle E_j| P |E_i\rangle$$
$$\int \left(\langle\Psi| P |\Psi\rangle\right)^2 d\mu_\Psi = \frac{1}{w(w+1)}\sum_i\sum_j \langle E_i| P |E_j\rangle \langle E_j| P |E_i\rangle + \frac{1}{w(w+1)} tr(\rho P)^2$$
$$\implies \langle\delta^2\rangle \leq \frac{1}{w(w+1)}\sum_i\sum_j \langle E_i| P |E_j\rangle \langle E_j| P |E_i\rangle$$
$$\langle\delta^2\rangle \leq \frac{1}{w(w+1)}\sum_i \langle E_i| P^2 |E_i\rangle = \frac{w}{w(w+1)} \quad \Big(\sum_j |E_j\rangle\langle E_j| \neq \mathbb{I};\ E_j \text{ is not a complete basis }\Big)$$
$$\langle\delta^2\rangle \leq \frac{1}{w+1}$$

Recall

$$w = e^S$$
$$\frac{1}{\sqrt{w}} = e^{-S/2} \tag{110}$$

This is a significant result!

Not only it is that the pure states on average look like microcanonical states but the average deviation of these states from the mixed states is exponentially suppressed. This result tell us that **for any observable most states almost look like the mixed state**

5 Lecture 8: The Old Information Paradox

We've not yet considered the back-reaction on geometry. But what we expect is that if the Black Hole is thermally populating these b-modes, it looses energy. The crudest version of the paradox is as follows: Start with matter in a pure state. Let it collapse and let the BH evaporate. SO it looks like we have Pure State $\longrightarrow$ Mixed State
For any pure state we have

$$\begin{aligned}
&\rho = |\Psi\rangle \langle\Psi| \\
&\rho^2 = \rho \\
&\rho(t) = U\rho U^\dagger \\
&\rho^2(t) = (U\rho U^\dagger)(U\rho U^\dagger) \\
&\rho^2(t) = \rho(t)
\end{aligned} \tag{111}$$

The answer to the above problem is the fact that we have just found the two point correlation functions $\left\langle b_{\omega k}\tilde{b}_{\omega' k'}\right\rangle$ are thermal. This does not imply that the final state is thermal
For example the following density matrix which looks thermal for a large class of observables A_α does in fact correspond to a pure state

$$tr(\rho_1 A_\alpha) = \frac{1}{Z} tr(e^{-\beta H} A_\alpha) + e^{-S/2} \tag{112}$$

5.1 Eigenstate Thermalization Hypothesis(ETH)

Observables that thermalize obey the ETH

$$\langle E_i| A_\alpha |E_j\rangle = A_\alpha(E)\delta_i j + e^{-S(\frac{E_i+E_j}{2})/2} R_{ij} \tag{113}$$

$S(\frac{E_i+E_j}{2})$: Density of states at $\frac{E_i+E_j}{2}$
R_{ij}: Random Phases

Now consider a state

$$|\Psi\rangle = \sum_E \frac{e^{-\beta E/2}}{\sqrt{Z(\beta)}} |E\rangle$$
$$\langle\Psi| A_\alpha |\Psi\rangle = \frac{1}{Z(\beta)} \sum_{E_i,E_j} e^{-\beta(E_i+E_j)/2} \langle E_i| A_\alpha |E_j\rangle$$
$$\langle\Psi| A_\alpha |\Psi\rangle = \frac{1}{Z(\beta)} \left[\sum_E e^{-\beta E} A_\alpha(E) + \sum_{E_i,E_j} e^{-S/2} e^{-\beta(E_i+E_j)/2} R_{ij}\right]$$

Magnitude of second term: For a given temperature β the total energy doesn't vary much and only a range of energies around some E_o is relevant. Also note that the number of eigenstates is by definition e^S and hence the summation runs upto e^{2S} terms. Now the sum of N random phases is of the order $\sqrt{N}$. So the second term is effectively of the order

$$\begin{aligned}
&\frac{1}{Z(\beta)} e^{-S/2} e^{-\beta E_o} e^S \\
=&e^{-S/2} \frac{e^{-\beta E_o + S}}{Z(\beta)} \\
=&e^{-S/2} \frac{e^{-\beta F}}{e^{-\beta F}} \\
=&e^{-S/2}
\end{aligned}$$

So we have proved

$$\langle\Psi| A_\alpha |\Psi\rangle = \frac{1}{Z(\beta)} tr(e^{-\beta H} A_\alpha) + \mathcal{O}(e^{-S/2}) \tag{114}$$

- Generic states $|\Psi\rangle$ behave thermally not just the one in the specific example above.

- **Most pure states look thermal for most observables** (upto exponential accuracy)

- The above generalization doesn't usually hold for vaccum state because it is not a generic state. Vacuum is a very special state. For the above generalization to hold the we need to consider states above a certain energy such that temperature can be defined for those states

5.2 Conclusions from the Old Information Paradox

- The fact that simple correlators behave thermally is far from sufficient to conclude that the state is mixed
- Hawking's calculation is not precise enough to lead to a paradox

Now many people tried to compute the corrections to Hawking's computation to check accurately if the final state is pure or mixed but this is futile!

Doubt in the reason

Tutorial Problems

(i) Compute entropy for the sun

Sol:

$$S = \frac{k_B A}{4l_p^2} \approx 10^{53} \tag{115}$$

(ii) Compute lifetime

Sol:

$$\begin{aligned} c^2\frac{dM}{dt} &= -\sigma 4\pi \left(\frac{2Ml_p}{m_p}\right)^2 \left(\frac{T_p m_p}{8\pi M}\right)^4 \\ \frac{dM}{dt} &= \frac{-\sigma(T_p^4 m_p^2 l_p^2)}{256\pi^3 M^2 c^2} \\ \implies \frac{M^3}{3} &= \frac{\sigma(T_p^4 m_p^2 l_p^2) t_{\text{life}}}{256\pi^3 c^2} \\ \implies t_{\text{life}} &= \frac{256\pi^3 M^3 c^2}{3\sigma(T_p^4 m_p^2 l_p^2)} = 10^{67} \text{ years} \end{aligned} \tag{116}$$

5.3 Page Curve

We can demand something more detailed than simply the fact that the final state is pure. Consider a region A "very far" from the Black Hole

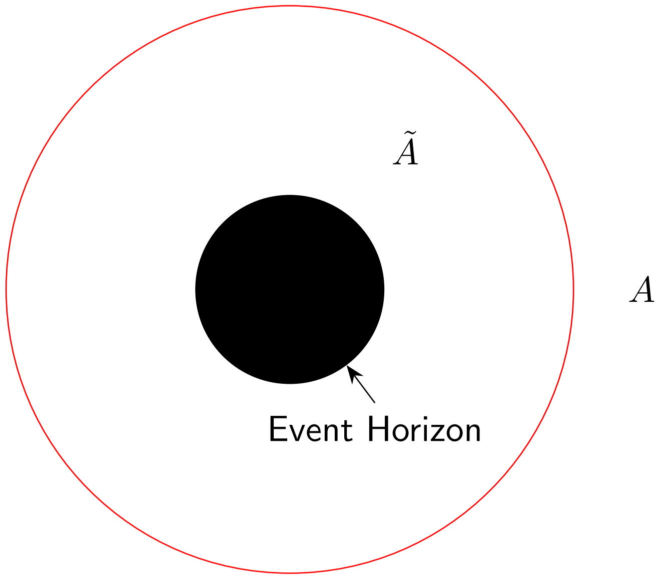

Figure 7

Define

$$\rho_A = tr_{\tilde{A}}(|\Psi\rangle \langle\Psi|) \tag{117}$$
$$S_A = -tr(\rho_A ln(\rho_A)) + S_o \tag{118}$$
$$S_o = tr(\rho_{vac} ln(\rho_{vac})) \tag{119}$$

$S_A \rightarrow$ Von Neumann entropy (a measure of how pure a state is)

S_A is a function of time. Don Page calculated this variation for a generic state and the S_A vs t curve so obtained is known as the Page Curve

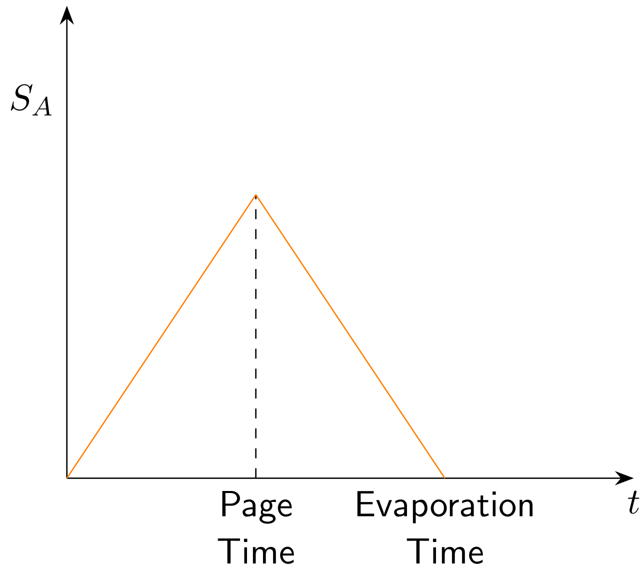

Figure 8: Page Curve

The curve can be roughly explained as follows: Initially $S_A = 0$ beacuse region A is unpopulated with any energy. As Hawking radiation begins to reach A S_A increases. But once the black hole has completely evaporated S_A should become 0 because the final state is a pure state and $S_A = 0$ in a pure state. So S_A which was increasing initially must have started to decrease at some point of time so that it eventually becomes 0. This time point is known as **Page Time**

Page argued that an evaporating black hole shoud follow the whole Page Curve rather than just the initial and final points.

6 Lecture 9: Modern Information Paradox

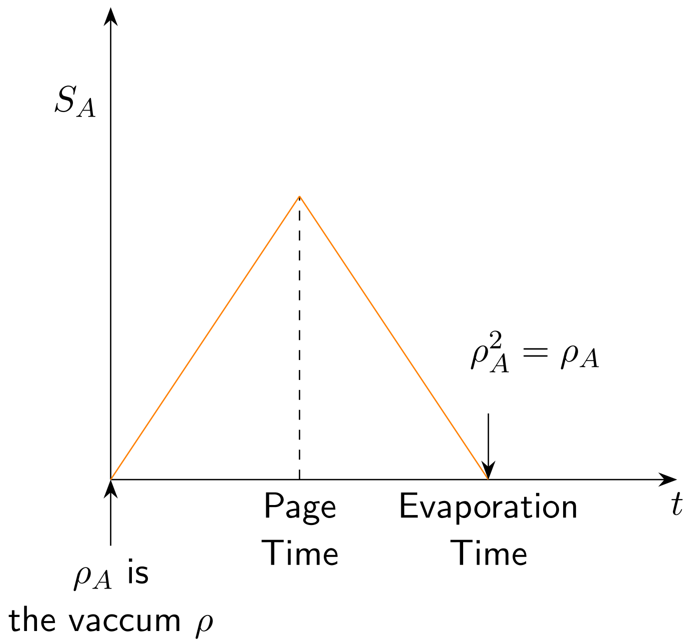

Figure 9: Page Curve

Note that the curve is not exactly linear with t but monotonically increases and decreases with t.

Mathur (2009) pointed out that this leads to a pardox. Consider three imaginary regions

Figure

$$
\begin{aligned}
&S_A(t+\delta t) = S_{A(t)\cup B(t)} = S_{AB} \\
&1) S_{AB} < S_A \quad \text{(After Page Time)} \qquad \text{(Unitarity and Genericity)} && (120)\\
&2) S_{BC} < S_B, S_C \qquad \text{(B and C are entangled)} && (121)\\
&3) S_B > 0, S_C > 0 \qquad \text{(Hawking Radiation is Thermal)} && (122)
\end{aligned}
$$

How to show B and C are entangled?

These three statements are in contradiction to a very general statement for three independent systems A,B and C

$$S_{AB} + S_{BC} \geq S_A + S_C \qquad \text{(Strong Subadditivity of Entropy)}$$

The paradox is known as SSE Paradox

The resolution that was proposed was that the interior region C is a Firewall or Fuzzball. So we drop the assumption $S_{BC} < S_C$. This can happend only if the correlators don't behave as expected classically (i.e. $\frac{1}{g^{\mu\nu}(y_{1_\mu}-y_{2_\mu})(y_{1_\nu}-y_{2_\nu})}$). This can happend if horizon is not smooth and there is infinite energy density at the horizon.

But the above conclusions violate effective field theory and should not be accepted unless we have no other option

Instead a better resolution would be to consider the fact that A,B and C are actually not independent and thus "Strong Subadditivity of Entropy" doesn't hold. Note that independence here means $[\phi_i, \phi_j] = 0$ for two systems i and j.

6.1 AMPSS Paradox

We would like to have

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{1}{1-e^{-\beta\omega}}$$

But we also expect

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{1}{Z}tr(e^{-\beta H}\tilde{b}_\omega\tilde{b}_\omega^\dagger) \qquad \text{(Equivalence of Ensembles and ETH Hypothesis)}$$

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{1}{Z}tr(\tilde{b}_\omega^\dagger e^{-\beta H}\tilde{b}_\omega)$$

$$\tilde{b}_\omega^\dagger e^{-\beta H} = e^{-\beta(H+\omega)}\tilde{b}_\omega^\dagger \qquad ([H,\tilde{b}_\omega^\dagger] = -\omega\tilde{b}_\omega^\dagger)\text{(- sign because modes inside the horizon)}$$

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{1}{Z}e^{-\beta\omega}tr(e^{-\beta H}\tilde{b}_\omega^\dagger\tilde{b}_\omega)$$

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{1}{Z}e^{-\beta\omega}tr(e^{-\beta H}(\tilde{b}_\omega\tilde{b}_\omega^\dagger - 1))$$

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{1}{Z}e^{-\beta\omega}\,\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle - e^{-\beta\omega}\frac{1}{Z}tr(E^{-\beta H})$$

$$\langle\Psi|\,\tilde{b}_\omega\tilde{b}_\omega^\dagger\,|\Psi\rangle = \frac{-e^{-\beta\omega}}{1-e^{-\beta\omega}} \longrightarrow \text{ABSURD!}$$

AMPSS resolution was a firewall proposal a/c to which interior doesn't exist and thus $\tilde{b}_\omega, \tilde{b}_\omega^\dagger$ don't exist.

We consider a construction of $\tilde{b}$ operators.

6.2 State Dependence

Take a bh state $|\Psi\rangle$

Recognise that EFT allows limited measurement (why?)

$$\langle\Psi| A_\alpha |\Psi\rangle$$

where A_α is a low-point polynomial in b_ω.

$$A_\alpha \in V$$

where V is the full set of observables a reasonable observer can measure

$$1)\, \langle\phi| (x_1) \cdots \phi(x_{10}) |\Psi\rangle \longrightarrow \text{Allowed}$$
$$2)\, \langle\phi| (x_1) \cdots \phi(x_S) |\Psi\rangle \longrightarrow \text{Not Allowed}$$

where S is the bh entropy

$$|\Psi\rangle = \text{energy state with the definitions}$$

$$\tilde{b}_\omega A_\alpha |\Psi\rangle = e^{-\beta\omega/2} A_\alpha b^\dagger |\Psi\rangle$$
$$\tilde{b}_\omega^\dagger A_\alpha |\Psi\rangle = e^{\beta\omega/2} A_\alpha b |\Psi\rangle$$

Imposing this $\forall A_\alpha$ gives dim(V) linear equations for $\tilde{b}_\omega$ but $\tilde{b}_\omega$ operates on a $e^S \times e^S$ space. So provided

$$dim(V) < e^S$$

we can solve these equations.

Note that this holds true under the assumption $A_\alpha |\Psi\rangle = 0 \forall A_\alpha \in V$

The additional relations that we can impose are

$$[H, \tilde{b}_\omega] = \omega \tilde{b}_\omega$$
$$[H, \tilde{b}_\omega^\dagger] = -\omega \tilde{b}_\omega^\dagger$$

Now let's check if the above constraints give us the expected commutation relations

$$
\begin{aligned}
[\tilde{b}_\omega, \tilde{b}^\dagger_\omega] |\Psi\rangle &= \left(\tilde{b}_\omega \tilde{b}^\dagger_\omega - \tilde{b}^\dagger_\omega \tilde{b}_\omega\right) |\Psi\rangle \\
&= \tilde{b}_\omega b_\omega |\Psi\rangle e^{\beta\omega/2} - \tilde{b}^\dagger_\omega b^\dagger_\omega |\Psi\rangle e^{-\beta\omega/2} && (\text{Put } A_\alpha = \mathbb{I}) \\
&= b_\omega b^\dagger_o |\Psi\rangle - b^\dagger_o b_\omega |\Psi\rangle \\
&= |\Psi\rangle
\end{aligned}
$$

An important observation that one should make in the above calculation is the fact that **these commutation relation hold about a given state of the in-falling observer. So this commutator isn't an identity operator. Just that it behaves as identity for low-point correlators**.

Now we evaluate one such two-point function

$$
\begin{aligned}
\langle\Psi| \tilde{b}_\omega \tilde{b}^\dagger_\omega |\Psi\rangle &= \langle\Psi| \tilde{b}_\omega b_\omega |\Psi\rangle e^{\beta\omega/2} && (\text{Put } A_\alpha = b_\omega) \\
&= \langle\Psi| b_\omega b^\dagger_\omega |\Psi\rangle \\
&= \frac{1}{1 - e^{\beta\omega}}
\end{aligned}
$$

$$
\begin{aligned}
\langle\Psi| \tilde{b}_\omega b_\omega |\Psi\rangle &= \langle\Psi| b_\omega b^\dagger_\omega |\Psi\rangle e^{-\beta\omega/2} && (\text{Put } A_\alpha = b_\omega) \\
&= \frac{e^{-\beta\omega/2}}{1 - e^{\beta\omega}} \longrightarrow VICTORY!
\end{aligned}
$$

The reason above equation is a victory goes as follows. We see that while calculating both the commutators and the two point functions we see there is a secret state dependence of $\tilde{b}_\omega$. This leads us to conclude

$$
\sum_i \langle\Psi_i| e^{-\beta H} \tilde{b}^{\dagger(\Psi)}_\omega \tilde{b}^{(\Psi)}_\omega |\Psi_i\rangle \neq \sum_i \langle\Psi_i| \tilde{b}^{\dagger(\Psi)}_\omega e^{-\beta H} \tilde{b}^{(\Psi)}_\omega |\Psi_i\rangle
$$

Basically for these state dependent $\tilde{b}^{\dagger(\Psi)}_\omega$ operators trace is not defined and thus the cyclicity of the trace is lost. This avoids the occupancy or AMPSS Paradox (6.1)!

Now we discuss the resolution of strong subadditivity paradox. First we compute the commutators of operators inside and outside the horizon to check if they actually point to B and C being independent.

$$\begin{aligned}[\tilde{b}_\omega, b_\omega] |\Psi\rangle &= \tilde{b}_\omega b_\omega |\Psi\rangle - b_\omega \tilde{b}_\omega |\Psi\rangle \\ &= b_\omega b_\omega^\dagger |\Psi\rangle e^{-\beta\omega/2} - b_\omega b_\omega^\dagger |\Psi\rangle e^{-\beta\omega/2} \\ &= 0\end{aligned}$$

Now we might think this implies B and C are independent. But this does not mean $[\tilde{b}_\omega, b_\omega] = 0$ or the commutator is 0 as an operator. This leads us to the conclusion that locality holds in low-pt correlators but not exactly and thus B and C are not independent. This resolves the Strong Subadditivity Paradox!