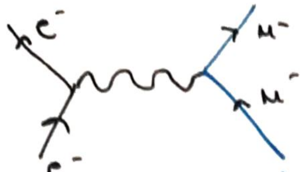
Radiative Corrections (scattering with heavy particle)

eg. $e^- + \mu^- / p^- \longrightarrow e^- + \mu^-/p^-$



(Tree-level)

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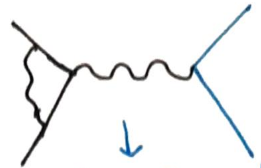

Vertex Correction

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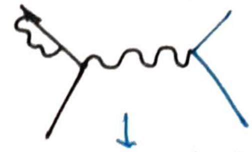

External Leg

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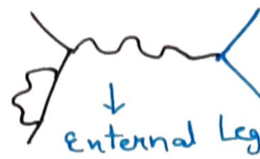

External Leg

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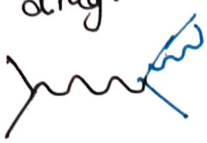
Vaccum Polarization

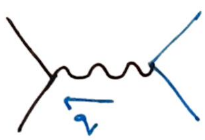
Note thate we don't include the loops with the heavy particle μ^-, since it doesn't contribute to the first order diagrams.

Reason :



→ (suppressed by mass of propogator) (in this case heavy particle)

(i) Vertex Correction



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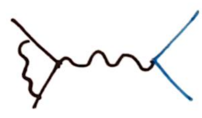

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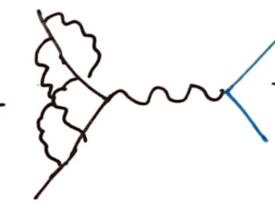

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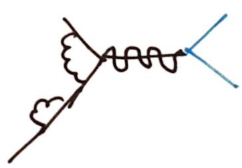
(We don't include this



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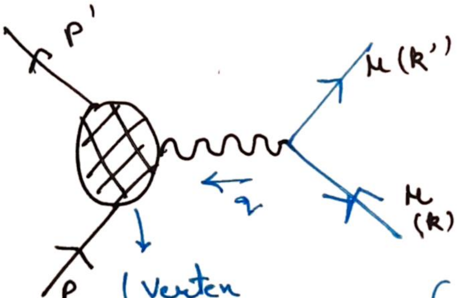


(Vertex correction) → Can be evaluated order by order using Feynman Rules

$$i\mathcal{M} = \left[\bar{u}(p')\, ie\Gamma_\nu u(p)\right] \frac{i\eta^{\mu\nu}}{q^2} \left[\bar{\mu}(k')\, ie\gamma_\mu\, \mu(k)\right]$$

↓

Vertex func^n → unknown

Try to guess Γ_ν using symmetries

$\Gamma^\mu = \gamma^\mu, p^\mu, p'^\mu, \not{p}, \not{p}', \underbrace{p^2, p'^2}_{\text{Lorentz scalars}}, m, e \rightarrow$ can depend on these quantities

$$\Gamma^\mu = \gamma^\mu A(p, p') + (p^\mu + p'^\mu) B(p, p') + (p^\mu - p'^\mu) C(p, p')$$

$A, B, C \rightarrow$ scalar functions

Ward Identity $q_\mu \Gamma^\mu = 0$ (will prove later)

$$\Rightarrow \not{q} A(p, p') + q\cdot(p+p') B(p,p') + q\cdot(p-p') C(p,p') = 0 \quad -(1)$$

$q = p' - p$

$q\cdot(p+p') = (p'-p)(p+p') = (p')^2 - (p)^2 = m^2 - m^2 = 0$

Consider $\bar{u}(p')\not{q}u(p) = \bar{u}(p')(\not{p}' - \not{p})u(p)$

(Dirac eq^n $\not{p}u(p) = m\,u(p)$
$\bar{u}(p)\not{p} = \bar{u}(p)\,m$)

$= \bar{u}(p')(m-m)u(p) = 0$

$q\cdot(p-p') = -q^2 \rightarrow$ (need not be zero for a virtual photon)

$\Rightarrow$ For (1) to be satisfied $C(p, p') = 0$

$$\Gamma^\mu = \gamma^\mu A(p, p') + (p^\mu + p'^\mu) B(p, p')$$

Gordon Identity $\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\dfrac{p'^\mu + p^\mu}{2m} + \dfrac{i\,\Sigma^{\mu\nu} q_\nu}{2m}\right] u(p)$

where $\Sigma^{\mu\nu} = \dfrac{i}{2}[\gamma^\mu, \gamma^\nu]$

Proof: $\bar{u}(p')\dfrac{i\sigma^{\mu\nu}q_\nu}{2m}u(p) = \dfrac{i}{2m}\cdot\dfrac{i}{2}\,\bar{u}(p')[\gamma^\mu, \gamma^\nu](p'_\nu - p_\nu)u(p)$

$$= -\frac{1}{4m}\bar{u}(p')\left[(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)p'_\nu - (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)p_\nu\right]u(p) \quad -(2)$$

③

$$\bar{u}(p')\gamma^\mu\gamma^\nu p_\nu u(p) = \bar{u}(p')\gamma^\mu \not{p}\, u(p) = \underline{m\,\bar{u}(p')\gamma^\mu u(p)}$$

$$\bar{u}(p')\, p'_\nu \gamma^\nu \gamma^\mu u(p) = m\,\bar{u}(p')\gamma^\mu u(p)$$

$$\Rightarrow \bar{u}(p')\, \frac{i\sigma^{\mu\nu}q_\nu}{2m}\, u(p) = \frac{1}{2}\bar{u}(p')\gamma^\mu u(p) - \frac{1}{4m}\left(\bar{u}(p')\gamma^\mu\gamma^\nu p'_\nu + \bar{u}(p')\gamma^\nu\gamma^\mu p_\nu u(p)\right]$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

$$\Rightarrow \bar{u}(p')\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\, u(p) = \frac{\bar{u}(p')\gamma^\mu u(p)}{2} - \frac{1}{4m}\left[2\bar{u}(p')\left(p'^\mu + p^\mu\right)u(p) - 2m\,\bar{u}(p')\gamma^\mu u(p)\right]$$

$$= \bar{u}(p')\gamma^\mu u(p) - \frac{\bar{u}(p')\left(p'^\mu + p^\mu\right)u(p)}{2m}$$

H.P.

Using the identity ~~$\bar{u}(p')$~~

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\Sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$$

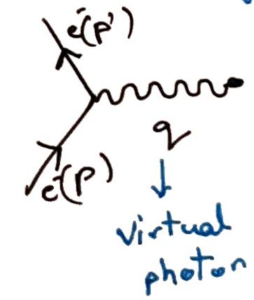
Form Factors → Goal: to calculate this

To lowest order $F_1(q^2) = 1$

$F_2(q^2) = 0$

Scattering of an e from External Mag. Field

$$H_{int} = \int d^3x\, e A^{cl}_\mu j^\mu \quad \text{where } j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$$



$$i\mathcal{M}\;\delta(p'_0 - p_0) = -ie\,\bar{u}(p')\gamma^\mu u(p)\,\tilde{A}_\mu(p'-p)$$

$i\mathcal{M}$ ↑ Feynman Amplitude

γ^μ ↓ Γ^μ (vertex correction)

$\tilde{A}_\mu$ ↓ FT of gauge field

$$A^{cl}_\mu(x) = (0, \vec{A}) \rightarrow \text{static vector pot.}$$

$$i\mathcal{M} = ie\left[\bar{u}(p')\left(\gamma^i F_1(q^2) + \frac{i\Sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right) u(p)\tilde{A}^i\right] \rightarrow (3)$$

Consider NR Limit $p', p, q \to 0$

$$A_\mu^{cl}(x) = (0, \vec{A}(\vec{x}))$$

$$\tilde{A}_\mu^{cl}(\vec{q})$$

$$u(k) = \frac{\not{k} + m}{\sqrt{2m(E+m)}} u(0) = \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\, \varphi(0) \\ \frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{2m(E+m)}}\, \varphi(0) \end{pmatrix}$$

$$\begin{cases} \varphi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & s = \frac{1}{2} \\ \varphi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & s = -\frac{1}{2} \end{cases}$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

First term of $\mathcal{M}$

$$\bar{u}(p')\gamma^i u(p) = u^\dagger(p')\gamma^0\gamma^i u(p) = u^\dagger(p') \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} u(p)$$

$$\bar{u}(p')\gamma^i u(p) = \left(\varphi^\dagger(0)\sqrt{\frac{E'+m}{2m}},\ \varphi^\dagger(0)\frac{\vec{\sigma}\cdot\vec{p}'}{\sqrt{2m(E'+m)}} \right) \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\, \varphi(0) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\, \varphi(0) \end{pmatrix}$$

$$= \frac{1}{2m}\left(\sqrt{\frac{E+m}{E'+m}}\, \varphi^\dagger(0)\, \vec{\sigma}\cdot\vec{p}'\, \sigma^i \varphi(0) + \sqrt{\frac{E'+m}{E+m}}\, \varphi^\dagger(0)\, \sigma^i\, \vec{\sigma}\cdot\vec{p}\, \varphi(0) \right)$$

NR limit $E = E' \cong m$

$$= \frac{1}{2m}\left(\varphi^\dagger(0)\, \vec{\sigma}\cdot\vec{p}'\, \sigma^i \varphi(0) + \varphi^\dagger(0)\, \sigma^i\, \vec{\sigma}\cdot\vec{p}\, \varphi(0) \right)$$

Use $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$

$$= \frac{1}{2m}\varphi^\dagger(0)\left[\sigma^j p'^j \sigma^i + \sigma^i \sigma^j p^j \right]\varphi(0)$$

$$= \frac{1}{2m} \varphi^{\dagger}(0) \Big[(p'+p)^i + i\epsilon^{jik} \sigma^k p'^j + \underbrace{i\epsilon^{ijk}\sigma^k p^j}\Big] \varphi(0)$$

$$i\epsilon^{jik}(p'^j - p^j)\sigma^k$$

$$= \cancel{\frac{1}{2m}}$$

$$= -i\epsilon^{ijk} q^j \sigma^k$$

Term linear in q^j $\rightarrow$ $\frac{-i}{2m} \varphi^{\dagger}(0)(\epsilon^{ijk} q^j \sigma^k)\varphi(0)$

Second term of $\mathcal{M}$

$$\frac{i}{2m} \bar{u}(p') \underbrace{\sigma^{i\nu} q_\nu}_{\sigma^{ij} q_j} u(p) = \frac{i}{2m} u^{\dagger}(p') \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} u(p) \epsilon^{ijk} (-q^j)$$

$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad \Big| \quad u(p) = \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\, \varphi(0) \\ \frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{2m(E+m)}}\, \varphi(0) \end{pmatrix}$$

Can set this to zero bcoz there's already a q present

$$\Rightarrow \frac{i}{2m} \varphi^{\dagger}(0) \frac{\sqrt{(E+m)(E'+m)}}{2m} \sigma^k \varphi(0) \epsilon^{ijk}(-q^j)$$

$$\downarrow$$

$$(\approx 1)$$

$$\Rightarrow \frac{-i}{2m} \varphi^{\dagger}(0)(\epsilon^{ijk} q^j \sigma^k)\varphi(0)$$

Subs. in (3) we get

$$i\mathcal{M} = ie\, \varphi^{\dagger}(0) \Big[\frac{-1}{2m} \sigma^k (F_1(0) + F_2(0)) \Big] \varphi(0) \left(i\epsilon^{ijk} q^j \tilde{A}^i_{cl}(\vec{q}) \right)$$

$$\boxed{\begin{aligned} &\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A} \quad \cancel{= \epsilon^{ijk}\partial_j A_k} \\ &B_i = \epsilon^{ijk}\partial_j A_k \xleftrightarrow{FT} \tilde{B}^i = -\epsilon^{ijk} i q_j \tilde{A}_k \\ &\qquad\qquad = -\epsilon^{kji} q_j \tilde{A}_i \\ &\qquad\qquad \tilde{B}^k = \epsilon^{ijk} q_j \tilde{A}_i \\ &\qquad\qquad \tilde{B}^k = \epsilon^{ijk} q^j \tilde{A}^i \end{aligned}}$$

$$i\mathcal{M} = ie\,(2m)\, \varphi^{\dagger}(0) \Big[\frac{-1}{2m} \sigma^k (F_1(0) + F_2(0)) \Big] \varphi(0) \tilde{B}^k(q)$$

$$V(\vec{r}) = -\langle\vec{\mu}\rangle \cdot \vec{B}(\vec{r})$$

$$\langle\vec{\mu}\rangle = \frac{e}{2m}\, 2\left[F_1(0)+F_2(0)\right] \underbrace{\varphi^{+}(0)\, \frac{\vec{\sigma}}{2}\, \varphi(0)}_{\vec{S}}$$

$$= \left(\frac{e}{2m}\right) g\, \vec{S}$$

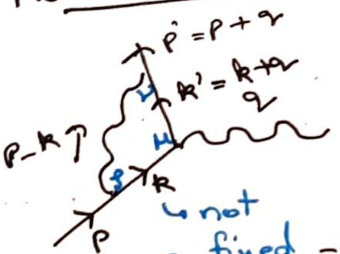
$$g = 2\left[F_1(0)+F_2(0)\right]$$

($\because F_1(q^2) = 1$ to lowest order, it won't receive any correction in first order)

Thus $\boxed{g = 2 + 2F_2(0)}$ → Anamolus mag. moment due to loop corrections

2 ↓ Dirac's Theory

Now we consider one loop correction



↳ not a fixed value
So need to integrate

$$\Gamma^{\mu} = \gamma^{\mu} + \delta\Gamma^{\mu}$$

$$\bar{u}(p')\delta\Gamma^{\mu} u(p) = \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p')(ie\gamma^{\nu})\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^{\mu}\underbrace{\left(\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\right)}_{\text{Fermion propogator}}(ie\gamma^{\rho})u(p)\cdot\left(\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\right)$$

$$\bar{u}(p')\delta\Gamma^{\mu} u(p) = \int \frac{d^4k}{(2\pi)^4}(i)^2(ie)^2(-i)\frac{\bar{u}(p')\gamma^{\nu}(\not{k}'+m)\gamma^{\mu}(\not{k}+m)\gamma_{\nu}u(p)}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)((k-p)^2+i\epsilon)}$$

$$N^{r} \rightarrow \left.\begin{array}{l} \gamma^{\nu}\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma_{\nu} \\ \gamma^{\nu}\gamma^{\alpha}\gamma^{\mu}\gamma_{\nu} \\ \gamma^{\nu}\gamma^{\mu}\gamma_{\nu} \end{array}\right\} \text{Terms like these}$$

$$\bar{u}(p')\delta\Gamma^{\mu}u(p) = 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\not{k}\gamma^{\mu}\not{k}' + m^2\gamma^{\mu} - 2m(k+k')^{\mu}\right]u(p)}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)((k-p)^2+i\epsilon)}$$

Notice $\frac{1}{AB} = \int_0^1 \frac{dx}{[xA+(1-x)B]^2} = \int \frac{dx}{[x(A-B)+B]^2}$

$$\frac{1}{AB} = \int_0^1 dx\,dy \frac{\delta(x+y-1)}{[xA+yB]^2}$$

$$\frac{1}{A_1 . A_2 \dots A_n} = \int_0^1 dx_1\, dx_2 \dots dx_n \frac{\delta(\Sigma x_i - 1)(n-1)!}{[x_1 A_1 + x_2 A_2 + \dots x_n A_n]^n}$$

$$\frac{1}{(k-p)^2+i\epsilon} \cdot \frac{1}{k'^2-m^2+i\epsilon} \frac{1}{k^2-m^2+i\epsilon} = \int_0^1 dx\,dy\,dz \frac{\delta(x+y+z-1)\,2}{D^3}$$

$$D = \cancel{x} z[(k-p)^2 + \cancel{\ }] + y(k'^2-m^2) + x(k^2-m^2) + \underbrace{(x+y+z)}_{=1 \text{ (inside integral)}} i\epsilon$$

$$= x(k^2-m^2) + y(k'^2-m^2) + z((k-p)^2 + \cancel{\ }) + i\epsilon$$

$$k'^2 = (k+q)^2 = k^2+q^2+2k.q$$

$$(k-p)^2 = k^2+p^2-2k.p.$$

$$= \underline{xk^2} - xm^2 + \underline{yk^2} + yq^2 + y(2k.q) - ym^2 + \underline{zk^2} + zp^2 - z(2k.p) + i.\epsilon$$

$$= k^2 + 2k(y.q - zp) + yq^2 + zp^2 - (x+y)m^2 + 2i\epsilon$$

Subs. $\cancel{k}$ $l = k + (yq - zp)$

$$l^2 = k^2 + 2k(yq-zp) + y^2q^2 + z^2p^2 - 2yqzp$$

$$l^2 - D = \cancel{k} - yq^2 - zp^2 + (x+y)m^2 - 2i\epsilon + y^2q^2 + z^2p^2 - 2yqzp$$

$$= y(y-1)q^2 + (z(z-1))p^2 - 2yzpq + (1-z)m^2 - i\epsilon$$

$$= -y(x+z)q^2 \cancel{- z(x+y)p^2} + (1-z)^2 m^2 - 2yzpq - i\epsilon$$

$$= -xyq^2 + (1-z)^2 m^2 - yzq^2 - 2yzpq - i\epsilon$$

Use $p' = p+q$
$(p')^2 = p^2+q^2+2p.q \Rightarrow q^2+2p.q = 0$

$$\ell^2 - D = \underbrace{-xyq^2 + (1-z)^2 m^2}_{\Delta} - i\epsilon$$

$$D = \ell^2 - \Delta + i\epsilon \qquad \ell = k + yq - zp$$

$$\Delta = -xyq^2 + (1-z)^2 m^2$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0 \rightarrow (\because D \text{ is even in } \ell) \quad -(4)$$

$$\& \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \underset{\text{const.}}{\cancel{\&}} \int \frac{d^4\ell}{(2\pi)^4} \frac{\eta^{\mu\nu}}{D^3} \cdot \left(\frac{1}{4}\ell^2\right) \quad -(5)$$

↳ obtained by taking trace

↓ Using these, we simplify the N^μ

$$N^\mu = \bar{u}(p') \left[\cancel{k}\gamma^\mu \cancel{k}' + m^2\gamma^\mu - 2m(k+k')^\mu \right] u(p)$$

$$k' = k + q$$

$$\ell = k + yq - zp$$

$$N^\mu = \bar{u}(p') \left[(\cancel{\ell} - y\cancel{q} + z\cancel{p})\gamma^\mu(\cancel{\ell} + \cancel{q} - y\cancel{q} + z\cancel{p}) + m^2\gamma^\mu - 4m(\ell - yq + zp)^\mu - 2mq^\mu \right] u(p)$$

$$N^\mu = \bar{u}(p') \left[\cancel{\ell}\gamma^\mu\cancel{\ell} + \cancel{\ell}\gamma^\mu(\cancel{q} - y\cancel{q} + z\cancel{p}) + (-y\cancel{q} + z\cancel{p})\gamma^\mu\cancel{\ell} + (-y\cancel{q} + z\cancel{p})\gamma^\mu(\cancel{q} - y\cancel{q} + z\cancel{p})^\mu + \ell^\mu 4m + 4m(yq^\mu - zp^\mu) - 2mq^\mu + m^2\gamma^\mu \right] u(p)$$

→ Terms linear in ℓ, goes go to zero inside integral

$$\Rightarrow N^\mu \text{(inside integral)} = \bar{u}(p') \left[\cancel{\ell}\gamma^\mu\cancel{\ell} + (-y\cancel{q} + z\cancel{p})\gamma^\mu(\cancel{q} - y\cancel{q} + z\cancel{p}) + m^2\gamma^\mu - 2m(q^\mu - 2yq^\mu + 2zp^\mu) \right] u(p)$$

$$\cancel{\ell}\gamma^\mu\cancel{\ell} = \ell_\alpha \ell_\beta \gamma^\alpha\gamma^\mu\gamma^\beta$$

$$= \ell_\alpha \ell_\beta (\gamma^\alpha 2\eta^{\mu\beta} - \gamma^\alpha\gamma^\beta\gamma^\mu)$$

$$= 2\ell_\alpha \ell^\mu \gamma^\alpha - \ell^2\gamma^\mu$$

$$\text{Use (5)} = 2\frac{1}{4}\delta^\mu_\alpha \ell^2 \gamma^\alpha - \ell^2\gamma^\mu = -\frac{\ell^2}{2}\gamma^\mu$$

$$\{\cancel{\ell}\cdot\cancel{\ell} = \ell^2\}$$

$$N^\mu = \bar{u}(p') \left[-\frac{1}{2}\gamma^\mu \ell^2 + (-y\cancel{q} + z\cancel{p})\gamma^\mu(\cancel{q} - y\cancel{q} + z\cancel{p}) + m^2\gamma^\mu - 2m(q^\mu - 2yq^\mu + 2zp^\mu) \right] u(p)$$

Goal : To show

$N^{\mu} \Rightarrow \bar{u}(p')\left[\cancel{\gamma^\mu - \frac{1}{2}\ell^2\gamma^\mu}\right.$

$$N^{\mu} \rightarrow \bar{u}(p')\left[\gamma^{\mu}\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\right) + m\,z(z-1)\,(p'+p)^{\mu} + m\,(z-2)(y-x)q^{\mu}\right]u(p)$$

$\boxed{(6)}$

$\rightarrow (-y\not{q} + z\not{p})\,\gamma^{\mu}(\not{q} - y\not{q} + z\not{p})$

Use $\bar{u}(p')\not{p}' = \bar{u}(p')\,m$, $x+y+z=1$

$\not{p}\,u(p) = m\,u(p)$, $p' = p+q$

$\bar{u}(p')\not{p}u(p) = \bar{u}(p')\,m\,u(p)$

$= (-y\not{q} + z\not{p}' - z\not{q})\,\gamma^{\mu}((1-y)\not{q} + mz)$

$= (-y\not{q} + z\not{p}' - z\not{q})\,\gamma^{\mu}((1-y)\not{q} + mz)$

$= ((x-1)\not{q} + mz)\,\gamma^{\mu}((1-y)\not{q} + mz)$

$= (x-1)(1-y)\,\underbrace{\not{q}\gamma^{\mu}\not{q}} + mz(x-1)\not{q}\gamma^{\mu} + mz(1-y)\gamma^{\mu}\not{q} + m^2z^2\gamma^{\mu}$

$\not{q}\gamma^{\mu}\not{q} = 2q^{\mu}\not{q} - \gamma^{\mu}q^2 = -\gamma^{\mu}q^2$ $\quad$ ($\because$ $\bar{u}(p')\not{q}u(p) = \bar{u}(p')(\not{p}'-\not{p})u(p) = 0$)

$= (1-x)(1-y)q^2\gamma^{\mu} + m^2z^2\gamma^{\mu} + mz(\cancel{1-y})^{(x-1)}[2m\gamma^{\mu} - 2p^{\mu}] + mz(1-y)[2q^{\mu} - 2m\gamma^{\mu} + 2p^{\mu}]$

$\bar{u}(p')(\not{p}'-\not{p})\gamma^{\mu}u(p)$

$\bar{u}(p')(m\gamma^{\mu} - \not{p}\gamma^{\mu})u(p)$ $\quad$ $\left\{\not{p}\gamma^{\mu} = 2p^{\mu} - \gamma^{\mu}\not{p}\right\}$

$\bar{u}(p')(m\gamma^{\mu} - 2p^{\mu} + \gamma^{\mu}\not{p})u(p)$

$\bar{u}(p')(2m\gamma^{\mu} - 2p^{\mu})u(p)$

$\Rightarrow \boxed{\not{q}\gamma^{\mu} = 2m\gamma^{\mu} - 2p^{\mu}}$

$= (1-x)(1-y)q^2\gamma^{\mu} + 2mz(1-y)q^{\mu} + 2mzp^{\mu}(1+z) + \gamma^{\mu}m^2(-z^2 - 2z)$

$m^2\gamma^{\mu} - 2m(q^{\mu} \mp 2yq^{\mu} + 2zp^{\mu})$

Remaining terms in $N^{\mu} \rightarrow$

$$N^{\mu} = \bar{u}(p')\left[\cancel{-\frac{1}{2}\gamma^{\mu}\ell^2}\,\gamma^{\mu}\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + m^2(1-z^2-2z)\right) + 2m\left[(1-y)z \mp \cancel{z(1+z)}\,(1-2y)\right]q^{\mu} + 2m\left[z^2 - z\right]p^{\mu}\right]$$

Consider the last two terms

$$2m\,[z^2-z]p^\mu = m(z)(z-1)[p^\mu+p'^\mu-q^\mu] = m\,z(z-1)(p+p')^\mu$$
$$-\,m\,q^\mu(z^2-z)$$

Combine with 2nd last term

$$m q^\mu\left[2z(1-y)-2(1-2y)-z^2+z\right]$$
$$2z-2yz-2+4y-z^2+z$$
$$2(2y+z-1)-z(z-1+2y) = (2-z)(2y+z-1)$$
$$=(2-z)(x-y)$$

Thus we get

N^γ ~~as~~ equal to (6)

Now using the Wald's identity contribution from q_μ terms is zero.

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu+p^\mu}{2m} + i\frac{\Sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

$$\frac{1}{2m}\bar{u}(p')(p'^\mu+p^\mu)u(p) = \bar{u}(p')\gamma^\mu u(p) - i\frac{\bar{u}(p')\Sigma^{\mu\nu}q_\nu u(p)}{2m}$$

Use this to modify N^γ & obtain correction to Form factors

$$\bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p) = \int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{N^\gamma}{D^\gamma}$$

$$N^\gamma \to \bar{u}(p')\Big[\gamma^\mu\Big(-\frac{1}{2}l^2+(1-x)(1-y)q^2+(1-2z-z^2)m^2\Big) + (p'^\mu+p^\mu)m\,z(z-1)\Big]u(p)$$

Use Gordon Identity

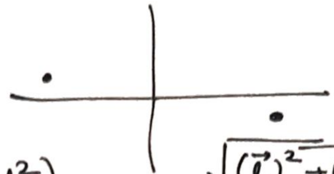
$$= \bar{u}(p')\Big[\gamma^\mu\Big(-\frac{1}{2}l^2+(1-x)(1-y)q^2+(1-4z+z^2)m^2\Big) + i\frac{\Sigma^{\mu\nu}q_\nu}{2m}.2m\,z(z-1)\Big]u(p)$$

$$\bar{u}(p')\,\delta\Gamma^\mu(p,p')\,u(p) = 2ie^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{D^3}.$$

$$\bar{u}(p')\Big[\gamma^\mu\Big(-\frac{l^2}{2}+(1-x)(1-y)q^2+(1-4z+z^2)m^2\Big) + i\frac{\Sigma^{\mu\nu}q_\nu}{2m}.2m(z)(z-1)\Big]u(p)$$

$l \to$ four vector

$D = l_0^2 - (\vec{l})^2 - \Delta + i\epsilon$ complex l^0 plane



Poles

$\sqrt{(\vec{l})^2 + \Delta - i\epsilon}$

Wick Rotation

$l_0 \to i l_0$

$l_E = (i l_0, l_i)$ $\quad (l_E^2 = -l^2)$ $\quad \hookrightarrow (l_0^2 - (\vec{l})^2 = -l_0^{E2} - (\vec{l}_E)^2 = -l_E^2)$

$$\bar{U}(p')\, \delta\Gamma^\mu(q^2)\, U(p) = 2ie^2 \int_0 dx\, dy\, dz\; i \int_0 \frac{d^4 l_E}{(2\pi)^4}$$

We need to perform two types of integrals

$$\boxed{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2-\Delta)^m}} \qquad \boxed{\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2-\Delta)^m}}$$

$m=3$ in this case

$$= \frac{i(-1)^m}{(2\pi)^4} \int d\Omega_3 \int_0^\infty dl_E \frac{l_E^3}{(l_E^2+\Delta)^m} \qquad \frac{i}{(-1)^m} \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2+\Delta)^m}$$

$d l_E^4 = l_E^3\, d\Omega_3$

$\hookrightarrow$ Area of unit three sphere $= 2\pi^2$

$$= \frac{i(-1)^m}{8\pi^2} \int_0^\infty dl_E\; l_E \frac{l_E^2}{(l_E^2+\Delta)^m}$$

$l_E^2 + \Delta = \alpha$

$d\alpha = 2 l_E\, dl_E$

$$= \frac{i(-1)^m}{16\pi^2} \int_\Delta^\infty d\alpha \frac{\alpha - \Delta}{(\alpha)^m} = \frac{i(-1)^m}{(4\pi)^2} \left[\frac{1}{(m-1)(m-2)\Delta^{m-2}}\right] \quad -(7)$$

$$\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2-\Delta)^m} = \frac{i(-1)^m}{4\pi^2} \left[\frac{2}{(m-1)(m-2)(m-3)} \cdot \frac{1}{\Delta^{m-3}}\right]$$

diverges at $m=3$

Note that this integral appears only in the first form factor $F_1(q^2)$

~~A possible rea~~

(The reason it diverges is larger values of l (or k) & it can be ultraviolet divergence traced back to photon propogator $\to$ introduce cutoff)

$$\frac{1}{(k-p)^2 + i\epsilon} \to \frac{1}{(k-p)^2 + i\epsilon} - \frac{1}{(k-p)^2 - \Lambda^2 + i\epsilon}$$

Unaffected for small k

Cuts off for $k \geq \Lambda$

$$\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2-\Delta)^3} \longrightarrow \int \frac{d^4 l}{(2\pi)^4} \left(\frac{l^2}{(l^2-\Delta)^3} - \frac{l^2}{(l^2-\Delta_\Lambda)^3} \right)$$

$$\Delta_\Lambda = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2$$

$$= \frac{i}{(4\pi)^2} \left[\log\left(\frac{\Delta_\Lambda}{\Delta}\right) + O(\Lambda^{-2}) \right]$$

$\delta F_1(q^2) \longrightarrow \delta F_1(q^2) - \delta F_1(0) \Rightarrow F_1(0) = 1$ (at one loop)

$$\int_0^1 dx\,dy\,dz \; \frac{1-4z+z^2}{\Delta(q^2=0)} \delta(x+y+z-1) \text{ (From (7))}$$

$$= \int_0^1 dx\,dy\,dz \; \delta(x+y+z-1) \frac{1-4z+z^2}{m^2(1-z)^2} = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-z-y} dx \; \frac{1-4z+z^2}{m^2(1-z)^2} \delta(x+y+z-1)$$

$$= \int_0^1 dz \int_0^{1-z} dy \; \frac{-2+(1-z)(3-z)}{m^2(1-z)^2} = \int_0^1 dz \; \frac{-2+(1-z)(3-z)}{m^2(1-z)}$$

$\boxed{\text{infrared divergence}}$ $\rightarrow$ again present in $F_1(q^2)$

No divergence present in $F_2(q^2)$

$$F_2(q^2) = \frac{\alpha}{2\pi} \int_0^1 dx\,dy\,dz \; \frac{2m^2 z(1-z)}{m^2(1-z)^2 - xyq^2} \delta(x+y+z-1)$$

$$F_2(q^2=0) = \frac{\alpha}{2\pi} \int_0^1 dx\,dy\,dz \; \frac{2m^2 z\cancel{(1-z)}}{m^2(1-z)^{\cancel{2}}} \delta(x+y+z-1)$$

$$= \frac{\alpha}{2\pi} \int_0^1 dz \int_0^{1-z} dy \int_0^{1-z-y} dx \; \frac{2z}{(1-z)} \delta(x+y+z-1)$$

$$\boxed{F_2(q^2=0) = \frac{\alpha}{2\pi} = \frac{g-2}{2} \simeq 0.0011614}$$

Electron Self Energy

p (propogator) at tree-level

$$= \frac{i(\not p + m)}{p^2 - m_0^2 + i\epsilon}$$

$\downarrow$ mass of $e^{\ominus}$

$+$

(Electron Self Energy)

$\downarrow$ will give correction to m_0

$$\frac{i(\not p + m_0)}{p^2 - m_0^2 + i\epsilon}\left(-i\,\Sigma_2(p)\right)\frac{i(\not p + m_0)}{p^2 - m_0^2}$$

$$-i\,\Sigma_2(p) = (-ie)^2 \int \gamma^\mu \frac{i(\not k + m_0)}{k^2 - m_0^2 + i\epsilon}\gamma_\mu \frac{(-i)}{(p-k)^2 - \mu^2 + i\epsilon}\frac{d^4k}{(2\pi)^4}$$

μ^2: small photon mass to avoid infrared div.

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}$$

$$\left(\frac{1}{k^2 - m_0^2 + i\epsilon}\right)\left(\frac{1}{(p-k)^2 - \mu^2 + i\epsilon}\right) = \int_0^1 dx \frac{1}{[k^2 - 2x(k\cdot p) + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}$$

$\downarrow$ Complete the square

$k^2 - 2x(k\cdot p)$ $\quad \cancel{k \to} k \quad \dfrac{1}{(l^2 - \Delta + i\epsilon)^2}$

$l = k - xp$

$\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$

$$N^\gamma = (-ie)^2(-i^2)\quad \gamma^\mu(\not k + m_0)\gamma_\mu$$

$$= -e^2\gamma^\mu(\not l + x\not p + m_0)\gamma_\mu$$

$\hookrightarrow$ terms linear in l would vanish from integral

$$= -e^2(-2x\not p + 4m_0)$$

$$-i\,\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^4 l}{(2\pi)^4}\frac{(-2x\not p + 4m_0)}{(l^2 - \Delta + i\epsilon)^2}$$

$\rightarrow$ divergent for large l

U.V. cutoff Λ

Pauli-Villiars Regularization

$$\frac{-i}{(p-k)^2 - \mu^2 + i\epsilon} \longrightarrow \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon} - \frac{(-i)}{(p-k)^2 - \Lambda^2 + i\epsilon}$$

$$\frac{1}{(\ell^2-\Delta)^2} \longrightarrow \frac{1}{(\ell^2-\Delta)^2} - \frac{1}{(\ell^2-\Delta_\Lambda)^2}$$

$\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$

$\Delta_\Lambda = -x(1-x)p^2 + x\Lambda^2 + (1-x)m_0^2$

$\xrightarrow{\Lambda\to\infty} x\Lambda^2$

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} \longrightarrow \frac{i}{(4\pi)^2}\int_0^\infty d(\ell_E^2)\left[\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} - \frac{(\ell_E^2+\Delta_\Lambda-\Delta_\Lambda)}{(\ell_E^2+\Delta_\Lambda)^2}\right]$$

$\ell^0 \to \ell^0_E = -i\ell^0$

$\log\left(\frac{\Delta_\Lambda}{\Delta}\right)$

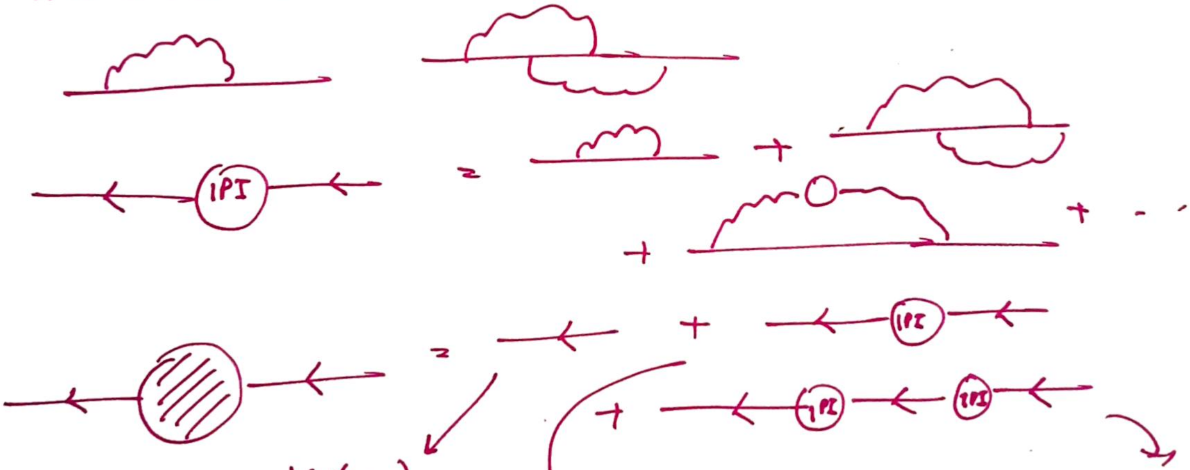
$$\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\log\left(\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right)$$

$$(1-x)m_0^2 + x\mu^2 - x(1-x)p^2 = 0$$

$$x = \frac{1}{2} + \frac{m_0^2-\mu^2}{2p^2} \pm R$$

Branch cut for sufficiently large p.

1PI → One-particle irreducible diagrams



$$\frac{i(\not{p}+m_0)}{p^2-m_0^2}$$

single pole at $p = m_0$

$$\frac{i(\not{p}+m_0)}{p^2-m_0^2}(-i\Sigma)\frac{i(\not{p}+m_0)}{(p^2-m_0^2)} + \cdots$$

Double pole at $p = m_0$

$$\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(p)}{\not{p}-m_0}\right) + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(p)}{\not{p}-m_0}\right)^2 + \cdots$$

$$= \frac{1}{\not p - m_0} \frac{1}{1 - \frac{\Sigma(p)}{\not p - m_0}} = \frac{1}{\not p - m_0 - \Sigma(p)}$$

$\downarrow$ enact propogator has a ~~po~~ shifted pole

The simple Pole is located

$$(\not p - m_0 - i\Sigma(\not p))|_{\not p = m} = 0$$

$\downarrow$ bare mass

$$\delta m = m - m_0 = \Sigma_2(\not p = m) \simeq \Sigma_2(\not p = m_0)$$

$$= \frac{\alpha}{2\pi} m_0 \int_0^1 dx \, (2-x) \log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)$$

$\downarrow$ shift in mass is divergent (logarithmic)

renormalize $\rightarrow$ (actually QED langrangian contains m_0 (bare mass) which itself is divergent & that's why δm_0 is div.)