Special Black Hole Geometries

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Outline

Killing Horizons

Reissner Nordström Solution

Kerr Solution

Penrose Process

Black Hole Thermodynamics

Killing Horizons

- If a killing vector field χ^μ is null along some null hypersurface Σ, then Σ is said to be a killing horizon of χ^μ
- For eg. in Minkowski space $\chi = x\partial_t + t\partial_x$ is killing vector field which generates boost in x-direction and it becomes null at null hypersurfaces $x = \pm t$. Hence these are killing horizons of χ
- To every event horizon we can assosciate a killing vector which is null at the horizon and so we can say every event horizon is a killing horizon for some killing vector field but not vice versa
- For eg. ∂_t is a killing vector field for Schwarzschild event horizon

RN Metric

- $ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$
- $\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$
- Asymptotically flat, spherically symmetric and static

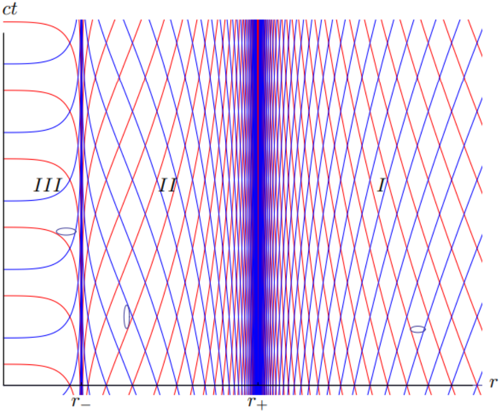
Singularities of the metric

- True(curvature) singularity at r=0
- Coordinate singularity(horizons) at $\Delta = 0$
- Setting $\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$ equal to 0 three possible cases arise
    1. $GM^2 > Q^2$ - Black Hole with two event horizons
    2. $GM^2 = Q^2$ - Extremal Black Hole
    3. $GM^2 < Q^2$ - Naked Singularity

$GM^2 > Q^2$ -Black Hole with two event horizons

- Two event horizons at $r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$
- Causal structure of radial light rays

$$0 = -\Delta dt^2 + \Delta^{-1}dr^2$$

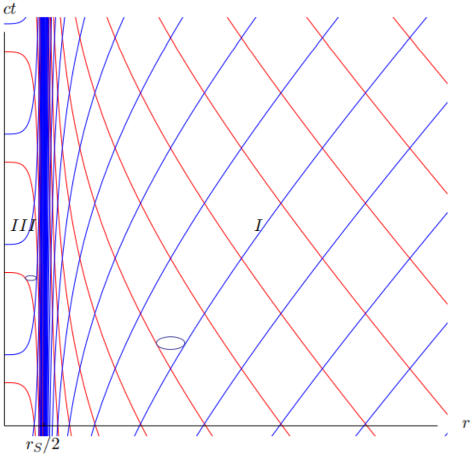


$GM^2 > Q^2$ -Black Hole with two event horizons

- In region I and III ∂_t is timelike and ∂_r is spacelike while in region II ∂_t is spacelikelike and ∂_r is timelike
- Since ∂_t is also a killing vector which is null at event horizons (null surfaces) so just like Schwarzschild case, here also the event horizons are killing horizons of the metric
- For an observer freely falling inside a RN black hole it can be proved that it never reaches the singularity at r=0
- The observer can manage to reach $t = \infty$ in a finite proper time and thus can escape from the black hole into another region of spacetime

$GM^2 = Q^2$ -Extremal Black Hole

- Not stable solutions as a slight charge may drive them to case I
- Casual structure of radial light rays (Absence of region II)



$GM^2 < Q^2$ -Naked Singularity

- If such a spacetime exist , then the singularity would be visible to the naked eyes of an observer at any radial coordinate
- But naked singularities arising from a gravitational collapse are usually forbidden in asymptotically flat spacetime . This is the famous Cosmic Censorship Conjencture formulated by Roger Penrose in 1969

Kerr metric for rotating black holes

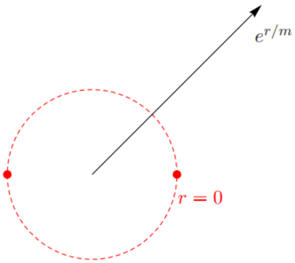
- $ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta^2} dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2$
- $a = \frac{J}{M}$; angular momentum per unit mass of black hole
- $\Delta = r^2 - 2GMr + a^2$
- $\rho^2 = r^2 + a^2\cos^2\theta$

Properties of Kerr metric

- Axisymmetric and asymptotically flat
- Staticity lost (due to a cross term $dtd\phi$)
- Stationary (No explicit time dependence of $g_{\mu\nu}$)
- The range of the Boyer-Lindquist coordinates of the Kerr metric with $a > 0$ is $t \in \mathbb{R}$, $r \in \mathbb{R}$, $(\theta, \phi) \in S^2$

Singularities of the metric

- True(curvature) singularity at $\rho = 0$
- Equating $r^2 + a^2 \cos^2\theta$ to 0 we get , the curvature singularity at $r^2 = 0$ and $\theta = \pi/2$
- Ring singularity



Singularities of the metric

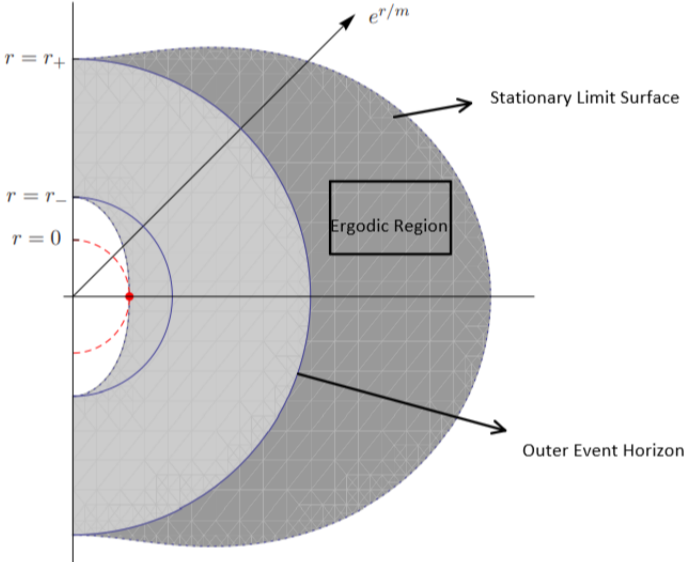
- Coordinate singularity(horizons) at $\Delta = 0$
- Setting $\Delta = r^2 - 2GMr + a^2$ equal to 0 three possible cases arise
  1. $GM^2 > a^2$ - Kerr Black Hole with two event horizons at $r_\pm = GM \pm \sqrt{GM^2 - a^2}$
  2. $GM^2 = a^2$ - Extremal Kerr Black Hole with event horizon at $r = GM$
  3. $GM^2 < a^2$ - Naked Singularity

Killing vectors of the metric

- As metric is not explicitly dependent on t and ϕ components, so $K = \partial_t$ and $R = \partial_\phi$ are two killing vector fields
- Unlinke Schwarzschild and R-N case, ∂_t is not null at event horizon, in fact it beccomes null at a value of $r > r_+$
- The surface at which ∂_t becomes null is given by

$$
\begin{aligned}
& g_{tt} = 0 \\
\implies & \rho^2 - 2GMr = 0 \\
\implies & r^2 - 2GMr + a^2 \cos^2\theta = 0
\end{aligned}
$$

Ergosphere


$e^{r/m}$
$r = r_+$
$r = r_-$
$r = 0$
Stationary Limit Surface
Ergodic Region
Outer Event Horizon

Ergosphere

- Outside ergoregion ∂_t is timelike , while it is spacelike inside ergodic region
- The outer boundary of ergodic region is known as Stationary Limit Surface because any observer inside the region can't remain at rest.
- However for a suitable constant $\Omega = \frac{d\phi}{dt}$ observer can manage to remain at a constant $r > r_+$
- At horizon $r = r_+$ there is a fixed value of $\Omega = \frac{a}{r_+^2 + a^2}$, also known as angular velocity of event horizon(Ω_H)

Killing vector associated with horizon

- Killing vector field for which the outer event horizon of a kerr black hole is a killing horizon is

$$\xi = \partial_t + \Omega_H \partial_\phi$$

- For $r > r_+$, $\partial_t + \Omega_H \partial_\phi$ is timelike while it is spacelike for $r < r_+$

Penrose Process

- For a freely falling particle the timelike geodesic $\gamma(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ has the tangent vector

$$\dot{\gamma} = \dot{t}\partial_t + \dot{r}\partial_r + \dot{\theta}\partial_\theta + \dot{\phi}\partial_\phi$$

(the derivatives are wrt τ)

- Total energy E of a particle moving along a geodesic(which is a constant of motion) is given by

$$\begin{aligned} E &= -g_{\mu\nu}p^\mu K^\nu \\ &= -g_{tt}p^t - g_{t\phi}p^\phi \\ &= m\left(1 - \frac{2GMr}{\rho^2}\right)\frac{dt}{d\tau} + \frac{2mGMar}{\rho^2}\sin^2\theta\frac{d\phi}{d\tau} \\ &= -m.g(\dot{\gamma}, \partial_t) \end{aligned}$$

Penrose Process

- $\dot{\gamma}$ is always timelike while ∂_t is timelike outside stationary limit surface and spacelike inside it. So E is necessarily positive outside ergoregion while it can be positive or negative inside it.
- Angular Momentum L of a particle moving along a geodesic(which is also a constant of motion) is given by

$$
\begin{aligned}
L &= g_{\mu\nu} p^{\mu} R^{\nu} \\
&= -g_{\phi\phi} p^{\phi} - g_{\phi t} p^{t} \\
&= \frac{m(r^2 + a^2)^2 - m\Delta a^2 \sin^2\theta}{\rho^2} \sin\theta \frac{d\phi}{d\tau} - \frac{2mGMar}{\rho^2} \sin^2\theta \frac{dt}{d\tau} \\
&= m.g(\dot{\gamma}, \partial_{\phi})
\end{aligned}
$$

Penrose Process

- As $\xi = \partial_t + \Omega_H \partial_\phi$ is timelike in ergoregion we have,

$$
\begin{aligned}
& g_{\mu\nu} p^\mu \xi^\nu < 0 \\
\implies & g_{\mu\nu} p^\mu (K^\nu + \Omega_H R^\nu) < 0 \\
\implies & -E + \Omega_H L < 0 \\
\implies & \boxed{L < \frac{E}{\Omega_H}}
\end{aligned}
$$

- This means if a particle has negative energy inside ergoregion, it must also have negative angular momentum

Penrose Process

- If we consider a particle with total energy E(> 0) freely falling into the ergoregion and decaying into two particles with energies $E_1(> 0)$ and $E_2(< 0)$ respectively inside the ergoregion then following two things are observed

  1. The particle with $E_2(< 0)$ can never come out of ergoregion as it has negative energy and it would ultimately be sucked by Black Hole
  2. The decaying process can be arranged such that the particle with energy $E_1(> 0)$ can come out of ergoregion and thus we have following energy balance

	Black Hole	Outside Particle
Initial Energy	E_o	E
Final Energy	$E_o + E_2 < E_o$	$E_1 > E$

Penrose Process

- This is how we can extract energy from a Kerr Blackhole
- As the particle has energy $E_2 < 0$, it also has an angular momentum $L_2 < 0$, so in the process both energy and angular momentum of blackhole keeps on decreasing and the ergoregion also starts getting contracted
- The process goes on until the angular momentum of blackholes becomes 0 and the blackhole becomes a Schwarzschild blackhole so that the ergoregion finally disapperars

Surface Gravity(κ)

- In Newtonian gravity, the surface gravity of a gravitating body is just the acceleration experienced at its surface by a test particle. For the Earth, the surface gravity is 9.8 m/s^2 .
- Similarly for an event horizon, surface gravity can be defined as acceleration of a static observer near the horizon (as measured by static observer at infinity).
- For a test particle at radius r_o in Schwarzschild spacetime , magnitude of four acceleration is given by

$$\sqrt{g_{\mu\nu}a^{\mu}a^{\nu}} = \frac{2GM}{2r_o^2\sqrt{1-\frac{2GM}{r_o}}}$$

$$\kappa = \lim_{r_o \to 2GM}\left(\sqrt{1-\frac{2GM}{r_o}}\right)\left(\frac{2GM}{2r_o^2\sqrt{1-\frac{2GM}{r_o}}}\right) = \frac{1}{4GM}$$

Surface Gravity(κ)

- Similarly we can find surface gravity of a kerr blackhole

$$\kappa = \frac{r_+ - m}{2mr_+} = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$$

$$(m = GM)$$

(M $\rightarrow$ Mass of Black Hole)
($r_+ \rightarrow$ Radius of outer event horizon) .

Area of event horizon

- For a kerr black hole, at outer event horizon surface line element ds is given by

$$\gamma_{ij} dx^i dx^j = ds^2 = (r_+^2 + a^2 \cos^2\theta) d\theta^2 + \left[\frac{(r_+^2 + a^2)^2 \sin^2\theta}{r_+^2 + a^2 \cos^2\theta} \right] d\phi^2$$

- This induced metric can be integrated at outer event horizon surface to obtain the area of event horizon

$$A = \int \sqrt{\gamma_{ij}} d\theta d\phi = \int (r_+^2 + a^2) \sin\theta d\theta d\phi$$

$$\boxed{A = 4\pi(r_+^2 + a^2)}$$

Setting up the Analogy

Ideal Gas		Kerr Black Hole
T	$\Longleftrightarrow$	$\kappa/2\pi$
P	$\Longleftrightarrow$	$-\Omega_H$
V	$\Longleftrightarrow$	GJ
U	$\Longleftrightarrow$	GM
S	$\Longleftrightarrow$	A/4

Analogy with Laws of Thermodynamics

Law	Ideal Gas	Kerr Black Hole
0^{th} Law	For a system in equilibrium the temperature is a constant	The surface gravity is a constant (on the horizon)
1^{st} Law	dU = TdS -PdV	$GdM = \kappa/8\pi dA + \Omega_H dJ$
2^{nd} Law	$\delta S \geq 0$	$\delta A \geq 0$

$\delta A \geq 0$ is the famous **Area Theorem** given by Stephen Hawking in the year 1971