Tutorial 4 Solutions Fourier Transform

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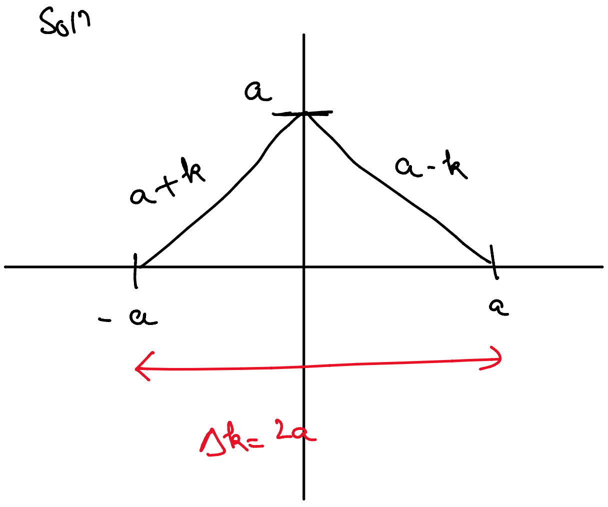
1. *If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
   (a) Find the Fourier transform for $\phi(k)$
   (b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

Soln



$$\int_{-\infty}^{\infty} \phi(k) \cdot \phi^*(k) \cdot dk = 1 \qquad \{\because \phi(k) \text{ is normalized}\}$$

$$\Rightarrow \int_{-a}^{0} A^2 (a+k)^2 . dk + \int_{0}^{a} A^2 (a-k)^2 . dk = 1$$

$$\Rightarrow \frac{A^2}{3}(a^3) + \frac{A^2}{3}(a^3) = 1$$

$$\Rightarrow \frac{2A^2}{3} a^3 = 1$$

$$\boxed{A = \pm\sqrt{\frac{3}{2}a^3}}$$

(Usually taken as +ve one)

(a) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \phi(k) e^{ikx} . dk = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} \underbrace{(k+a) e^{ikx} . dk}_{\text{Subs. } k = -k} + \int_{0}^{a} (a-k) e^{ikx} dk \right]$$

$$\hookrightarrow \int_{a}^{0} (-k+a) e^{-ikx} (-dk) = \int_{0}^{a} (a-k) e^{-ikx} dk$$

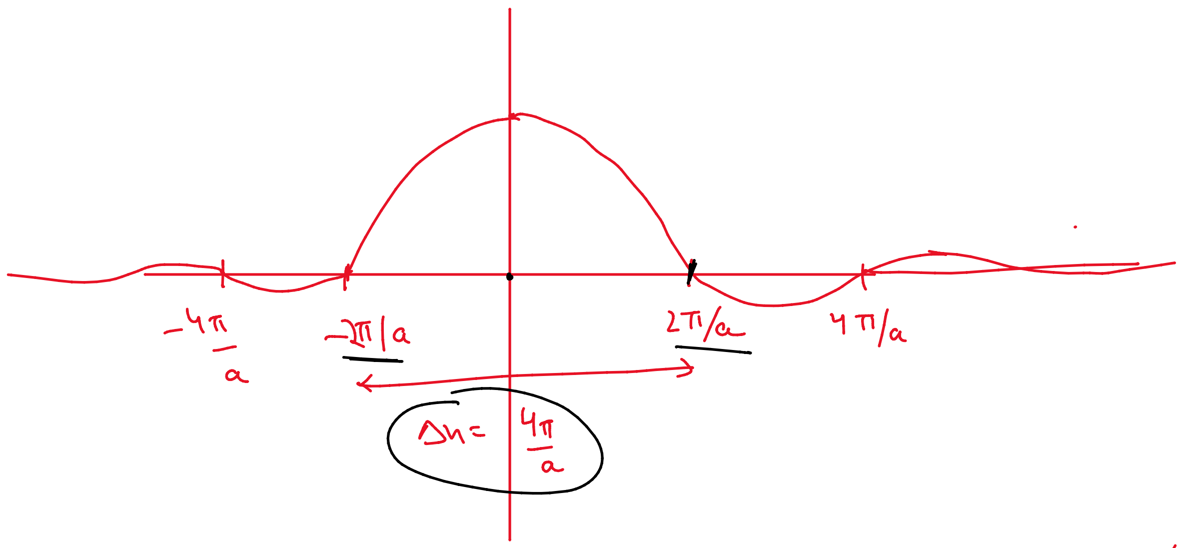
-

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{a} (a-k) (e^{ikx} + e^{-ikx}) \right] . dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} 2(a-k) \cos kx . dk$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(a-k)}{x} \sin kx \Big|_{0}^{a} + \int_{0}^{a} \frac{\sin kx . dk}{x} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(1 - \cos ax)}{x^2} \right]$$



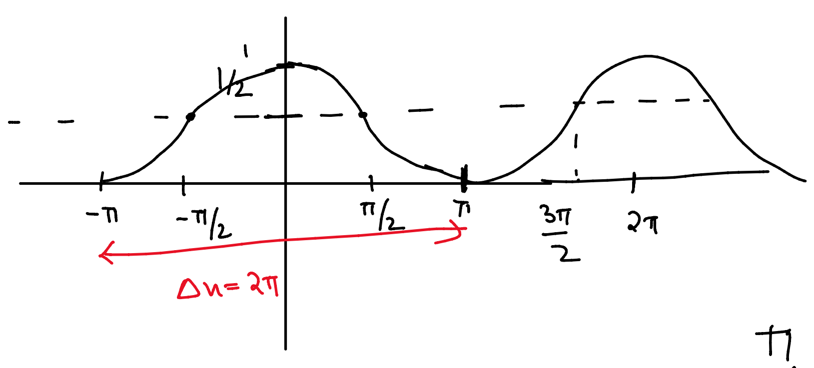
$p = \hbar k \Rightarrow \Delta p = \hbar \Delta k$

$\Delta x \cdot \Delta k = \frac{4\pi}{a} \cdot 2a = 8\pi = \Delta x \cdot \frac{\Delta p}{\hbar}$

$\boxed{\Delta x \cdot \Delta p = 8\pi\hbar}$

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)$ $(for -\pi \le x \le \pi)$ and $f(x) = 0$ elsewhere
   (a) Plot $f(x)$ versus x.
   (b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
   (c) At what value of k, $|g(k)|$ attains its maximum value?
   (d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
   (e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

(a)



$\cos^2(x/2) = \frac{1}{2}(1+\cos x)$

Sol^n (b)

$$g(k) = \int_{-\pi}^{\pi} \cos^2\left(\frac{x}{2}\right) e^{-ikx}\,dx = \int_0^{\pi} \cos^2\frac{x}{2}\left(e^{-ikx} + e^{ikx}\right) dx$$

$$= \int_0^{\pi} \frac{(1+\cos x)}{\cancel{2}} (\cancel{2}\cos kx)\,dx$$

$$= \int_0^{\pi} \cos kx + \frac{1}{2}\left(\cos(k+1)x + \cos(k-1)x\right) dx$$

$$= \frac{\sin k\pi}{k} + \frac{\sin(k+1)\pi}{2(k+1)} + \frac{\sin(k-1)\pi}{2(k-1)}$$

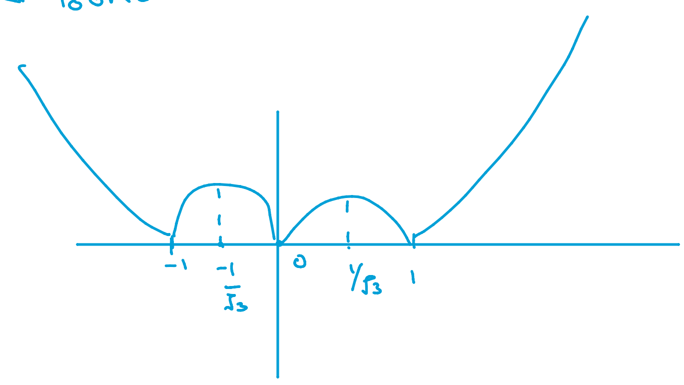
$$= \frac{\sin k\pi}{k} - \frac{1}{2}\left[\frac{\sin k\pi}{(k+1)} + \frac{\sin k\pi}{(k-1)}\right]$$

$$= \sin k\pi \left[\frac{(k^2-1) - (k^2)}{k(k^2-1)}\right]$$

$$= \frac{-\sin k\pi}{k(k^2-1)} = \frac{\sin(k\pi)}{k-k^3} = g(k)$$

(c) $|g(k)| = \left|\frac{\sin(k\pi)}{k-k^3}\right| = \frac{|\sin k\pi|}{|k-k^3|}$

(c) $|g(k)| = \left|\frac{\sin(k\pi)}{k-k^3}\right| = \frac{|\sin \ldots|}{(|k-k^3|)}$ looks like this

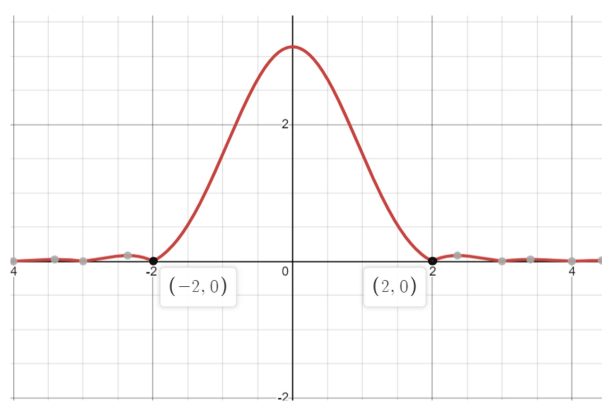


$\lim\limits_{k\to 0} \frac{-\sin k\pi}{k(k-1)(k+1)} = +\pi$

$\lim\limits_{k\to 1} -\frac{\sin k\pi}{k(k-1)(k+1)} = \frac{\sin(k-1)\pi}{k(k-1)(k+1)} = +\frac{\pi}{2}$

$\lim\limits_{k\to -1} -\frac{\sin k\pi}{k(k-1)(k+1)} = \frac{+\sin(k+1)\pi}{k(k-1)(k+1)} = +\frac{\pi}{2}$

since $|k-k^3|$



(d) Zeroes when $\sin k\pi = 0$ $(k \neq 0, 1, -1)$

$\Rightarrow k = \pm 2, \pm 3 \ldots$

$\Delta k = 4$

$\Delta x \cdot \Delta k = 8\pi$

Particle in a box

3. *A wave function $\psi(x)$ is defined such that $\psi(x) = \sqrt{2/L}\sin(\pi x/L)$ for $0 \le x \le L$ and $\psi(x) = 0$ otherwise.
(a) Writing $\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$, find $a(k)$.
(b) What is the amplitude of the plane wave of wavelength L constituting $\psi(x)$?

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(k)e^{ikx}\,dk$$
$$a(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(x)e^{-ikx}\,dx$$

Sol^n $a(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x)\,e^{-ikx}\,dx$

$$= \frac{1}{2\pi}\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^L\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\left(\cos kx - i\sin(kx)\right)dx$$

$$= \frac{1}{2\pi}\sqrt{\frac{2}{L}}\int_0^L \frac{1}{2}\left(\sin\left(\frac{\pi}{L}+k\right)x\right) + \frac{1}{2}\left(\sin\left(\frac{\pi}{L}-k\right)x\right)dx$$

$$+ \frac{1}{2\pi}\sqrt{\frac{2}{L}}(-i)\int_0^L \frac{1}{2}\cos\left(\frac{\pi}{L}-k\right)x - \frac{1}{2}\cos\left(\frac{\pi}{L}+k\right)x\,dx$$

$$= \frac{1}{2\pi}\sqrt{\frac{2}{L}}\,\frac{1}{2}\left[\frac{1+\cos kL}{\frac{\pi}{L}+k} + \frac{1+\cos kL}{\frac{\pi}{L}-k}\right] - \frac{i}{2\pi}\sqrt{\frac{2}{L}}\cdot\frac{1}{2}\left[\frac{\sin kL}{\frac{\pi}{L}-k} + \frac{\sin kL}{\frac{\pi}{L}+k}\right]$$

$$\frac{1}{\ldots}\left[\frac{(1+\cos kL)(2\pi)}{\ldots}\right] - i\left[\frac{(\sin kL)(2\pi)}{\ldots}\right]$$

$$= \frac{1}{2\pi}\sqrt{\frac{L}{2}}\left[\frac{(1+\cos kL)(2\pi)}{(\pi^2-k^2L^2)}\right] - i\left[\frac{(\sin kL)(2\pi)}{(\pi^2-k^2L^2)}\right]$$

$$= \sqrt{\frac{L}{2}}\,\frac{1}{(\pi^2-k^2L^2)}\left[(1+\cos kL) - i\,(\sin kL)\right]$$

corresponding amplitude

(b) $\quad f(x) \longleftrightarrow \sum_k a(k)\, e^{ikx}$

plane wave with wave number k

$$a\left(k=\frac{2\pi}{L}\right) = \sqrt{\frac{L}{2}}\left(\frac{1}{-3\pi^2}\right)(2)$$